Musterlösung Serie 20

Gram-Schmidt, Orthogonality

1. Let V be a finite dimensional euclidean vector space and $S \subset V$ an orthnormal set. Show that we can extend S to an orthomormal basis of V.

   Solution: Write $S = \{v_1, \ldots, v_m\}$ and extend $(v_1, \ldots, v_m)$ to an ordered basis $(v_1, \ldots, v_n)$ of V. Applying Gram-Schmidt yields an orthomnormal basis $(b_1, \ldots, b_n)$. The consruction implies $(b_1, \ldots, b_m) = (v_1, \ldots, v_m)$; thus $\{b_1, \ldots, b_n\}$ is an orthonormalbasis of V, which contains S.

   Aliter: By assumption S is an orthonormal basis $U := \langle S\rangle$. As V is finite dimensional, we get $V = U \oplus U^\perp$. Here $U^\perp$ is again finite dimensional and thus admits an orthonormal basis T. We saw in the lecture that $S \cup T$ is then an orthonormal basis of V.

2. Let $A \in M_{n\times n}(\mathbb{R})$. Show:

   (a) The matrix A^TA is symmetric.

   (b) The matrix A^TA is positive definite if and only if A is invertible.

   (c) We have $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$.

   (d) Assume that A is symmetric and that it admits $(\lambda_1, v_1), (\lambda_2, v_2)$ with $\lambda_1, \lambda_2 \neq 0$ as pairs of eigenvalue-eigenvector. Show that if $\lambda_1 \neq \lambda_2$ then $v_1 \perp v_2$.

   Solution: Assertion (a) follows from $(A^TA)^T = A^T(A^T)^T = A^TA$.

   Moreover, we have for every $v \in \mathbb{R}^n$

   $$(*) \qquad v^T \cdot (A^TA) \cdot v = (v^TA^T) \cdot Av = (Av)^TAv = \|Av\|^2,$$

   where $|| \; ||$ is the standard Euclidean norm on $\mathbb{R}^n$.

   If A is invertible, we get $Av \neq 0$ and hence $v^T \cdot (A^TA) \cdot v = \|Av\|^2 > 0$; so A^TA is positive definite. If on the other hand A^TA is positive definite, we get $\|Av\|^2 = v^T \cdot (A^TA) \cdot v > 0$ and hence $Av \neq 0$. So the linear map $L_A \colon \mathbb{R}^n \to \mathbb{R}^n$, $v \mapsto Av$ has trivial kernel, and hence A is invertible. Thus we proved (b).

   For (c) we start by claiming $\operatorname{Ker}(L_A) = \operatorname{Ker}(L_{A^TA})$.

   Proof: For all $v \in \mathbb{R}^n$ with $Av = 0$, we have $A^TAv = A^T0 = 0$; thus „$\subset$“. Conversely, let $A^TAv = 0$. By $(*)$, we get $\|Av\|^2 = v^T \cdot (A^TA) \cdot v = v^T0 = 0$. As the

Euclidean norm is positive definite, we get $Av = 0$, hence $v \in \operatorname{Ker}(L_A)$. This shows „$\supset$“. *q.e.d.*

The claim yields

$$\begin{aligned}\operatorname{Rang}(A) &= \dim \operatorname{Im}(L_A) = n - \dim \operatorname{Ker}(L_A) \\ &= n - \dim \operatorname{Ker}(L_{A^TA}) = \dim \operatorname{Im}(L_{A^TA}) = \operatorname{rank}(A^TA).\end{aligned}$$

To show (d), we compute that on one hand

$$\langle Av_1, Av_2 \rangle = \langle \lambda_1 v_1, \lambda_2 v_2 \rangle = \lambda_1 \lambda_2 \langle v_1, v_2 \rangle,$$

and on the other hand,

$$\langle Av_1, Av_2 \rangle = v_1^t A^t A v_2 = v_1^t A^2 v_2 = \lambda_2^2 v_1^t v_2 = \lambda_2^2 \langle v_1, v_2 \rangle,$$

where we used that A is symmetric to obtain the second equality. Hence

$$\lambda_2(\lambda_2 - \lambda_1)\langle v_1, v_2 \rangle = 0 \implies v_1 \perp v_2$$

since we assumed that $\lambda_1 \neq \lambda_2$ and that $\lambda_1, \lambda_2 \neq 0$.

3. Compute a decomposition $A = QR$ of the matrix

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

into an orthogonal matrix Q and an upper triangular matrix R. Use this decomposition to solve the linear system $Ax = b$ for $b = (0, 3, -3)^T$.

Lösung: Gram-Schmidt-Orthogonalisierungsverfahren yields the desired matrices

$$Q = \begin{pmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix} \quad \text{und} \quad R = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplication of $Ax = QRx = b$ on the left with the invertible matrix $Q^{-1} = Q^T$ yields the equivalent system of equations $Rx = Q^Tb = (4, 1, -1)^T$. As R is an upper triangular matrix, one quickly obtains the solution $x = (2, 1, -1)^T$.

4. Let U be the subspace of $\mathbb{R}^3$ with the standard inner product spanned by the two vectors $v_1 = (1, 1, 1)^T$ and $v_2 = (0, 2, 1)^T$.

   (a) Determine an orthonormal basis of U and an orthonormal basis of $U^\perp$.

   (b) Compute the representation matrices of the orthogonal projections $\mathbb{R}^3 \to U$ and $\mathbb{R}^3 \to U^\perp$ with respect to the standard basis of $\mathbb{R}^3$ and the bases from (a).

Lösung:

(a) A simple way of finding orthomal basis of U and $U^\perp$ simultaneously is to apply Gram-Schmidt on the vectors v_1, v_2 and an additional vector $v_3 \in \mathbb{R}^3$, which supplements them to a basis, e.g. $v_3 = (1,0,0)^T$. This yields

$$w_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad w_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}, \qquad w_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}.$$

Hence (w_1, w_2) is an orthonomal basis of U and w_3 one of U^T.

Alternatively, one uses Gram-Schmidt on (v_1, v_2) to compute an orthonormal basis (w_1, w_2) of U,and solves the system of linear equations $\langle v_1, w\rangle = \langle v_2, w\rangle = 0$ to compute a basis u' of U^T, such that $w_3 := u'/\|u'\|$ is an orthonormal basis of U^T.

(b) The orthogonal projection of $\mathbb{R}^3$ onto U is given by

$$v \mapsto \langle w_1, v\rangle w_1 + \langle w_2, v\rangle w_2.$$

The representation matrix with respect to the bases (e_1, e_2, e_3) of $\mathbb{R}^3$ and (w_1, w_2) of U is

$$\big(\langle e_j, w_i\rangle\big)_{\substack{1\leqslant i\leqslant 2\\ 1\leqslant j\leqslant 3}} = (w_1, w_2)^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}.$$

Analogously, one obtains the representation matrix of the orthogonal projection of $\mathbb{R}^3$ onto $U^\perp$

$$\big(\langle e_j, w_3\rangle\big)_{1\leqslant j\leqslant 3} = w_3^T = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.$$

5. For each of the following vector spaces V endowed with the inner product $\langle\cdot,\cdot\rangle$, find $U^\perp$ for the given subset U:

(a) First consider

$$V = \left\{(a_0, a_1, a_2, \dots) \mid \sum_{n=0}^{\infty} |a_n|^2 < \infty\right\}, \quad \langle (a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty\rangle = \sum_{n=0}^{\infty} a_n \cdot b_n,$$

$$U = \{(a_n)_{n=0}^\infty \in V \mid \exists N \geqslant 0 \text{ s.t. } \forall m \geqslant N : a_m = 0\}$$

(b) Secondly, we set

$$V = C([0,1]), \quad \langle f, g\rangle = \int_0^1 f(x)\cdot g(x)dx,$$

$$U = \left\{f \in V \mid \int_0^{1/2} f(x)dx = 0\right\}.$$

Solutions:

(a) For $i \in \mathbb{Z}_{\geqslant 0}$, we denote $s^{(i)}$ the sequence whose elements all vanish, except for the element at index i, which equals 1. Note that for all $i \in \mathbb{Z}_{\geqslant 0}$, $s^{(i)} \in U$. Let $b = (b_n)_{n \in \mathbb{Z}_{\geqslant 0}} \in U^{\perp}$. By definition, for any $i \in \mathbb{Z}_{\geqslant 0}$, we then have

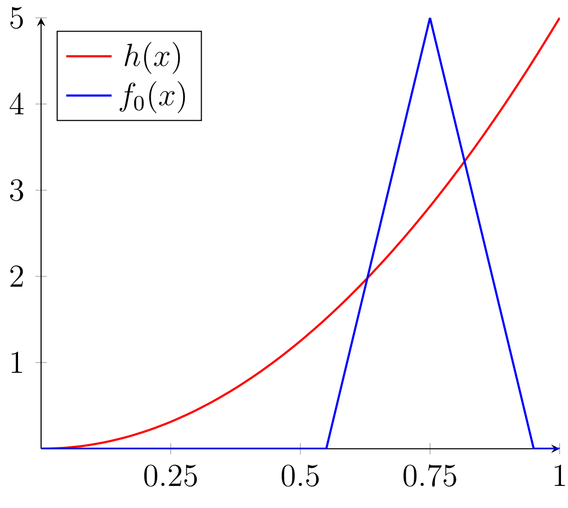
$$\sum_{n=0}^{\infty} s_n^{(i)} b_n = 0 \quad \Leftrightarrow \quad b_i = 0.$$

Since this holds for all indices i, it follows that $b = 0_V$. Since b was an arbitrary element of $U^{\perp}$, we conclude that $U^{\perp} = \{0_V\}$.

(b) First note that any function $h \in U^{\perp}$ must vanish on $[1/2, 1]$. Indeed, assume that some $h \in U^{\perp}$ doesn't vanish on $[1/2, 1]$. Then by continuity of h there exists $x_0 \in (1/2, 1)$ and $n \in \mathbb{N}$ big enough such that h doesn't change sign on the interval $(x_0 - 1/n, x_0 + 1/n)$ contained in $(1/2, 1)$. Let f_0 denote the piecewise linear function defined by

$$f_0(x) = \begin{cases} 0, & x \in \left[0, x_0 - \frac{1}{n}\right] \\ n, & x = x_0 \\ 0, & x \in \left[x_0 + \frac{1}{n}, 1\right] \end{cases}$$

Example for $h(x) = 5x^2$, $x_0 = 3/4$, $n = 5$.



Note that $f_0 \in U$. Then

$$\left| \int_0^1 f_0(x) h(x) dx \right| = \left| \int_{x_0 - 1/n}^{x_0 + 1/n} f_0(x) h(x) dx \right| > 0,$$

since the integrand doesn't change sign on the interval $\left[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right]$. This is a contradiction to $h \in U^{\perp}$.

We now show that any $g \in U^{\perp}$ must be constant on $[0,1/2]$. Fix $a,b \in (0,1/2)$ such that $a < b$. There exists $N(a,b) \in \mathbb{N}$ such that for all integers $n \geqslant N(a,b)$, the piecewise linear function f_n defined as follows belongs to V:

$$f_n(x) = \begin{cases} 0, & x \in \left[0, a - \frac{1}{n}\right] \\ n, & x = a \\ 0, & x \in \left[a + \frac{1}{n}, b - \frac{1}{n}\right] \\ -n, & x = b \\ 0, & x \in \left[b + \frac{1}{n}, 1/2\right] \end{cases}$$

Note that for f_n integrates to 0 on $[0,1/2]$, hence $f_n \in U$. Let $g \in U^{\perp}$. Then,

$$\int_0^{1/2} f(x)g(x)dx = 0 \quad \Leftrightarrow \quad \int_{a-1/n}^{a+1/n} f_n(x)g(x) = -\int_{b-1/n}^{b+1/n} f_n(x)g(x)dx.$$

Letting $n \to +\infty$, the second equality above converges to $g(a) = g(b)$. We show convergence to $g(a)$ for the left0-hand side of the said equality (the RHS can be handled similarly). Since g is continuous in a, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |g(x) - g(a)| < \varepsilon.$$

So, for n big enough so that $1/n < \delta$, we have

$$\begin{aligned} & \left| \int_{a-1/n}^{a+1/n} f_n(x)g(x)dx - \int_{a-1/n}^{a+1/n} f_n(x)g(a)dx \right| \\ = & \left| \int_{a-1/n}^{a+1/n} f_n(x)(g(x) - g(a))dx \right| \\ \leqslant & \int_{a-1/n}^{a+1/n} f_n(x) \left| g(x) - g(a) \right| dx \\ < & \ \varepsilon \int_{a-1/n}^{a+1/n} |f_n(x)| \, dx \\ = & \ \varepsilon, \end{aligned}$$

where we used that f_n integrates to 1 on $[a - 1/n, a + 1/n]$ to obtain the last equality. Letting ε go to 0, this shows the desired convergence.

Using the continuity of g and the fact that a and b are arbitrary, we deduce that g is constant on $[0,1/2]$. Since g vanishes on $[1/2,1]$, it follows, again from continuity, that g vanishes on the whole interval $[0,1]$. This shows that $U^{\perp} = \{0_V\}$.

6. For a finite-dimensional Euclidean vector space $(V, \langle\ ,\ \rangle)$ consider the isomorphism

$$\delta \colon V \to V^* := \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}), \ v \mapsto \delta(v) := \langle v, \cdot \rangle.$$

(a) Show that there exists exactly one inner product $\langle\ ,\ \rangle^*$ on V^* such that δ is an isometry.

(b) Let B be an ordered basis of V, and let B^* be the corresponding dual basis of V^*. Give the representation matrix of $\langle\ ,\ \rangle^*$ with respect to B^* in terms of the representation matrix of $\langle\ ,\ \rangle$ with respect to B.

Solution:

(a) The map δ is an isometry for the sought scalar product $\langle\, , \rangle^*$ if and only if

$$\forall v, w \in V\colon\ \langle \delta(v), \delta(w)\rangle^* = \langle v, w\rangle\,. \tag{1}$$

As δ is bijective, there exists exactly one map $\langle\, , \rangle^*$ with that property. Thus define $\langle\, , \rangle^*$ by the relation (1) or, equivalently by

$$\langle \lambda, \mu\rangle^* := \langle \delta^{-1}(\lambda), \delta^{-1}(\mu)\rangle$$

for all $\lambda, \mu \in V^*$. Linearity and injectivity of δ^{-1} implies that this is a scalar product on V^* with the desired property.

(b) Let $B = (v_1, \ldots, v_n)$ be an ordered basis of V and

$$A \ :=\ [\langle\, , \rangle]_B \ =\ \big(\langle v_i, v_j\rangle\big)_{i,j}$$

the representation matrix $\langle\, , \rangle$ with respect to B. Then $\delta(B) := (\delta(v_1), \ldots, \delta(v_n))$ is an ordered basis of V^*, and by construction the representation matrix of $\langle\, , \rangle^*$ with respect to $\delta(B)$ is

$$[\langle\, , \rangle^*]_{\delta(B)} \ =\ \big(\langle \delta(v_i), \delta(v_j)\rangle^*\big)_{i,j} \ =\ \big(\langle v_i, v_j\rangle\big)_{i,j} \ =\ A.$$

Now let $B^* = (v_1^*, \ldots, v_n^*)$ be the dual basis of B. For all j and k we then have

$$\Big(\sum_i \langle v_j, v_i\rangle\, v_i^*\Big)(v_k) \ =\ \sum_i \langle v_j, v_i\rangle\, v_i^*(v_k) \ =\ \sum_i \langle v_j, v_i\rangle\, \delta_{i,k} \ =\ \langle v_j, v_k\rangle \ =\ \delta(v_j)(v_k).$$

Variation of k yields

$$\delta(v_j) \ =\ \sum_i \langle v_j, v_i\rangle\, v_i^*.$$

Hence the base change matrix between $\delta(B)$ and B^* is

$${}_{B^*}[\mathrm{id}_V]_{\delta(B)} \ =\ \big(\langle v_j, v_i\rangle\big)_{i,j} \ =\ A^T \ =\ A.$$

Its inverse is the base change matrix

$${}_{\delta(B)}[\mathrm{id}_{V^*}]_{B^*} \ =\ (A^T)^{-1}.$$

The base formula from the lecture yields

$$\begin{aligned}[\langle\, , \rangle^*]_{B^*} \ &=\ {}_{\delta(B)}[\mathrm{id}_{V^*}]_{B^*}^T \cdot [\langle\, , \rangle^*]_{\delta(B)} \cdot {}_{\delta(B)}[\mathrm{id}_{V^*}]_{B^*} \\ &=\ ((A^T)^{-1})^T A A^{-1} \ =\ A^{-1} A A^{-1} \ =\ A^{-1}.\end{aligned}$$

Single Choice. In each exercise, exactly one answer is correct.

1. Consider the vector space $\mathbb{R}^2$ endowed with the standard scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|| \; ||$. For which vectors $v, w \in \mathbb{R}^2$ do we have $||v + w|| = ||v|| + ||w||$?

   (a) $v = \binom{1}{1}, \; w = \binom{1}{0}$

   (b) $v = \binom{2}{1}, \; w = \binom{1}{2}$

   (c) $v = \binom{2}{3}, \; w = \binom{3}{1}$

   $\boxed{\text{(d)}}$ $v = \binom{1}{2}, \; w = \binom{2}{4}$

   Erklärung: Equality holds if and only if one of the vectors is a non-negative multiple of the other; hence only (d) is correct. An alternative way of seeing this would be direct calculations.

2. Let V be a euclidean vector space and let $v_1, v_2, v_3 \in V$. Which assertion does in general not hold?

   $\boxed{\text{(a)}}$ From $v_1 \perp v_2$ and $v_2 \perp v_3$ it follows that $v_1 \perp v_3$.

   (b) From $v_1 \perp v_2$ and $v_1 \perp v_3$ it follows that $v_1 \perp (v_2 + v_3)$.

   (c) From $v_1 \perp v_2$ it follows that $v_1 \perp -v_2$.

   (d) From $v_1 \perp (v_2 + v_3)$ and $v_1 \perp v_2$ it follows that $v_1 \perp v_3$.

   Erklärung: For $v_1 = v_3 \neq 0 = v_2$ assertion (a) is false. The other assertions follow from the scalar product being bilinear.

3. Let V be a euclidean vector space and let $S, T \subset V$ be two subsets. Which of the following properties is not equivalent to the other three?

   (a) $S \subset T^\perp$

   (b) $T \subset S^\perp$

   (c) $S \perp T$

   $\boxed{\text{(d)}}$ $\langle S \rangle \cap \langle T \rangle = \{0\}$

   Erklärung: We have $S \perp T$ if and only if S is contained in $T^\perp = \{v \in V \mid v \perp T\}$. Hence (a) is equivalent to (c) and (b). However, the subsets $S := \{\binom{1}{0}\}$ and $T := \{\binom{1}{1}\}$ of $V = \mathbb{R}^2$ are not orthogonal to each other, but satisfy condition (d).

4. Let S be a subset of a finite dimensional euclidean vector space V. Which asstertion does not hold in general?

   (a) $(S^\perp)^\perp = \langle S \rangle$.

   [(b)] S is the orthogonal complement of a subspace of V.

   (c) $S^\perp$ is a subspace of V.

   (d) $V = S^\perp \oplus (S^\perp)^\perp$.

   Erklärung: The orthogonal complement of a subspace of V is always a subspace. If S is not a subspace, assertion (b) is false. The other assertions were proved in the lecture.

Multiple Choice Questions

1. Which of the following matrices are hermitian?

   [(a)] $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

   (b) $\frac{1}{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   (c) $\frac{1}{\sqrt{5}}\begin{pmatrix} i & -2 \\ 2 & i \end{pmatrix}$

   [(d)] $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

   Explanation: Check $A^* = A$.

2. Which of the following matrices are unitary?

   (a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

   [(b)] $\frac{1}{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   [(c)] $\frac{1}{\sqrt{5}}\begin{pmatrix} i & -2 \\ 2 & i \end{pmatrix}$

   [(d)] $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

   Explanation: Check $U^*U = \mathbf{1}$, equivalently: the columns form an orthonormal basis.

3. Let U, V be unitary $n \times n$ matrices, and let $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. In general, which of the following statements hold?

   (a) $U + V$ is unitary.

   Explanation: Wrong. $-U$ is unitary. $0 = U + (-U)$ is not.

   [(b)] λU is unitary.

   Explanation: True. As $|\lambda| = 1$, we have $\overline{\lambda} = \lambda^{-1}$. Hence $(\lambda U)^* = \overline{\lambda}U^* = \lambda^{-1}U^{-1} = (\lambda U)^{-1}$.

   [(c)] U^{-1} is unitary.

   Explanation: True. $(U^{-1})^* = (U^*)^{-1} = (U^{-1})^{-1} = U$.

   [(d)] UV is unitary.

   Explanation: True. $(UV)^* = V^*U^* = V^{-1}U^{-1} = (UV)^{-1}$.