Musterlösung Serie 22

Self-adjoint operators, Spectral theory

1. Let K be a field and let $(V, \langle\cdot,\cdot\rangle)$ be an inner product finite-dimensional K-vector space. Suppose that $T \in \mathrm{End}(V)$ and that U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

   Solution: Assume that U is invariant under T. Then, for any $u \in U$, for any $w \in U^\perp$,

   $$0 = \langle Tu, w\rangle = \langle u, T^*w\rangle.$$

   Hence for all $w \in U^\perp$, $T^*w \in U^\perp$.

   Conversely, assume that $U^\perp$ is invariant under T^*. Let $w \in U^\perp$ and let $u \in U$. We have

   $$0 = \langle u, T^*w\rangle = \langle Tu, w\rangle.$$

   Hence, for all $u \in U$, $Tu \in (U^\perp)^\perp = U$. Here we used that V is finite-dimensional.

2. Let f, g_1, and g_2 be endomorphisms of a finite-dimensional Euclidean vector space such that

   $$f^* \circ f \circ g_1 = f^* \circ f \circ g_2\,.$$

   Prove that $f \circ g_1 = f \circ g_2$.

   Lösung: For all $v, w \in V$, we have

   $$\begin{aligned}\Big\langle f(v), f\big((g_1 - g_2)(w)\big)\Big\rangle &= \Big\langle v, f^* \circ f\big((g_1 - g_2)(w)\big)\Big\rangle \\ &= \langle v, (f^* \circ f \circ g_1 - f^* \circ f \circ g_2)(w)\rangle \\ &= 0.\end{aligned}$$

   With $v := (g_1 - g_2)(w)$, it follows for all $w \in V$

   $$\|f\big((g_1 - g_2)(w)\big)\|^2 = 0$$

   and hence $f\big((g_1 - g_2)(w)\big) = 0$ and thus $(f \circ g_1)(w) = (f \circ g_2)(w)$.

3. Let V be a finite-dimensional unitary vector space and let $T \in \mathrm{End}(V)$ be a normal operator. For a subspace $W \subseteq V$, we denote P_W the orthogonal projection onto W.

   (a) Show the following:

Theorem. *There exist finitely many complex numbers* $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, *and mutually orthogonal subspaces of* V *denoted* $W_1, \ldots, W_k$ *such that*

$$T = \lambda_1 P_{W_1} + \cdots + \lambda_k P_{W_k}.$$

(b) Show that for any subspace U of V, P_U is self-adjoint.

Solution:

(a) By the spectral theorem for unitary vector spaces, T is orthogonally diagonalisable. For $j = 1, \ldots, k$, denote λ_j the eigenvalues of T, and let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of eigenvectors of T such that for integers $1 \leqslant \ell_1 < \ell_2 < \cdots < \ell_k = n$, we have

$$\begin{aligned}\mathrm{Eig}_T(\lambda_1) &= \mathrm{Sp}(v_1, \ldots, v_{\ell_1}),\\ \mathrm{Eig}_T(\lambda_j) &= \mathrm{Sp}(v_{\ell_{j-1}+1}, \ldots, v_{\ell_j}), \quad j = 2, \ldots, k.\end{aligned}$$

We show that setting $W_j = \mathrm{Eig}_T(\lambda_j)$ for $j = 1, \ldots, k$ proves the theorem. Indeed, since $\mathcal{B}$ is an orthogonal basis, the W_j's are automatically mutually orthogonal. Now consider $v \in V$ with $v = \sum_{i=1}^n a_i v_i$. We compute that

$$\begin{aligned}T(v) &= T\left(\sum_{i=1}^n a_i v_i\right)\\ &= \sum_{i=1}^{\ell_1} a_i T(v_i) + \cdots + \sum_{i=1}^{\ell_k} a_i T(v_i)\\ &= \lambda_1 \sum_{i=1}^{\ell_1} a_i v_i + \cdots + \lambda_k \sum_{i=\ell_{k-1}+1}^{\ell_k} a_i v_i.\end{aligned}$$

To conclude, note that, since $\mathcal{B}$ is orthonormal, we have $a_i = \langle v, v_i\rangle$. Hence each of the sums in the last line above equals $\lambda_j P_{W_j}(v)$, for $j \in \{1, \ldots, k\}$. This shows the desired equality.

(b) Let $\{u_1, \ldots, u_r\}$ be an orthonormal basis of U. We have $P_U^* = P_U$ if and only if for all $v, w \in V$ we have

$$\langle P_U(v), w\rangle = \langle v, P_U(w)\rangle.$$

We compute that

$$\begin{aligned}
\langle v, P_U(w)\rangle &= \left\langle v, \sum_{i=1}^{r} \langle w, u_i\rangle w \right\rangle \\
&= \sum_{i=1}^{r} \overline{\langle w, u_i\rangle} \langle v, u_1\rangle \\
&= \left\langle \sum_{i=1}^{r} \langle v, u_i\rangle u_i, w \right\rangle \\
&= \langle P_U(v), w\rangle.
\end{aligned}$$

4. Make $\mathbb{R}[x]_2$ into an inner product space by defining

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx$$

Define $T \in \operatorname{End}(\mathbb{R}[x]_2)$ by $T(a_0 + a_1 x + a_2 x^2) = a_1 x$.

(a) Show that T is not self-adjoint.

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

Solution:

(a) If T were self-adjoint, we would have

$$\langle Tp, q\rangle = \langle p, T^* q\rangle = \langle p, Tq\rangle.$$

However, let $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$. We have

$$\begin{aligned}
\langle Tp, q\rangle &= \langle a_1 x, q\rangle \\
&= a_1 \int_0^1 b_0 x + b_1 x^2 + b_2 x^3 dx \\
&= a_1 \left(\frac{b_0}{2} x^2 + \frac{b_1}{3} x^3 + \frac{b_2}{4} x^4 \right) \Bigg|_0^1 \\
&= a_1 \left(\frac{b_0}{2} + \frac{b_1}{3} + \frac{b_2}{4} \right).
\end{aligned}$$

Similarly

$$\langle p, Tq \rangle = \langle p, b_1 x \rangle = b_1 \left(\frac{a_0}{2} + \frac{a_1}{3} + \frac{a_2}{4} \right)$$

Thus, taking $a_1 = 0$ and $b_1, a_0, a_2 > 0$ clearly shows $\langle Tp, q \rangle \neq \langle p, Tq \rangle$.

(b) You have seen in the lectures that an endomorphism T of a real vector space with inner product $(V, \langle \cdot, \cdot \rangle)$ is orthogonally diagonalizable if and only if it is self-adjoint. The above does not contradict the forward implication since the basis $\{1, x, x^2\}$ is not an orthogonal basis of $\mathbb{R}[x]_2$ with respect to the inner product defined above.

5. Let $f : V \to W$ be a homomorphism of euclidian vector spaces.

   (a) Assume that $\dim(V) < \infty$. Show that the adjoint of f exists in the following sense: show that there exists a unique map $f' : W \to V$ such that

   $$\text{for all } v \in V, \text{ for all } w \in W : \langle f(v), w \rangle = \langle v, f'(w) \rangle.$$

   (b) Does the statement still hold if instead of assuming $\dim(V) < \infty$ we assume that $\dim(W) < \infty$?

Lösung:

(a) For a linear map $f : V \to W$, with V assumed to be finite-dimensional, the adjoint of f has been defined in the lectures as the map

$$\begin{array}{rccc} f^* : & W^* & \to & V^* \\ & \varphi & \mapsto & \varphi \circ f \end{array}$$

You have also seen that there exist linear maps

$$\begin{array}{rccc} \Phi : & V & \to & V^* \\ & v & \mapsto & \varphi_v \text{ s.t. } \varphi_v(v') = \langle v', v \rangle \text{ for all } v' \in V \end{array}$$

$$\begin{array}{rccc} \Psi : & W & \to & W^* \\ & w & \mapsto & \psi_w \text{ s.t. } \psi_w(w') = \langle w', w \rangle \text{ for all } w' \in W \end{array}$$

and that Φ is an isomorphism between V and V^* since V is finite-dimensional. We define $f' = \Phi^{-1} \circ f^* \circ \Psi$, as illustrated in the following diagram

$$\begin{array}{ccc} W^* & \xrightarrow{f^*} & V^* \\ {\scriptstyle \Psi}\uparrow & & \downarrow{\scriptstyle \Phi^{-1}} \\ W & & V \end{array}$$

Now, we compute that for any $w \in W$,

$$f'(w) = (\Phi^{-1} \circ f^* \circ \Psi)(w) = (\Phi^{-1} \circ f^*)(\psi_w)$$
$$= \Phi^{-1}(\psi_w \circ f)$$
$$= v_0,$$

where $v_0 \in V$ is such that for all $v \in V$,

$$\langle v, v_0 \rangle = \varphi_{v_0}(v) = (\psi_w \circ f)(v) = \langle f(v), w \rangle,$$

i.e. $\langle v, f'(w) \rangle = \langle f(v), w \rangle$.

Assume that another such map $h : W \to V$ exists. Then, for all $v \in V$, for all $w \in W$,

$$\langle v, f'(w) \rangle = \langle f(v), w \rangle = \langle v, h(w) \rangle,$$

which implies $f'(w) = h(w)$.

(b) No. For a counterexample let I be any infinite set and consider $V := \mathbb{R}^{(I)}$ endowed with the scalar product

$$\langle \underline{x}, \underline{y} \rangle \ := \ \sum_{i \in I} x_i y_i.$$

This space has the orthonormal basis $\{\underline{e_i} \mid i \in I\}$ with $\underline{e_i} = (\delta_{ij})_j$. Let $W := \mathbb{R}$ with the standard scalar product $\langle u, v \rangle := uv$, and consider the homomorphism

$$f \colon V \to W, \quad \underline{x} \mapsto \sum_{i \in I} x_i.$$

Assume that its adjoint $f^* \colon W \to V$ exists. Then $f^*(1) = (x_i)_{i \in I}$ with almost all $x_i = 0$. For every i we have

$$x_i \ = \ \langle \underline{e_i}, (x_i)_i \rangle \ = \ \langle \underline{e_i}, f^*(1) \rangle \ = \ \langle f(\underline{e_i}), 1 \rangle \ = \ \langle 1, 1 \rangle \ = \ 1.$$

Together, this is a contraditction; hence f^* does not exist.

6. Consider the vector space V consisting of all infinitely differentiable periodic functions from $\mathbb{R}$ to $\mathbb{R}$ with period 2π, equipped with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) g(x) dx.$$

Let $D : V \to V$ be the linear transformation defined by $D(f) = \frac{df}{dx}$.

(a) Is D self-adjoint? Determine its adjoint if it exists.

(b) Is $\Delta := -D \circ D$ self-adjoint?

(c) Let $U \subset V$ be the linear span of the functions

$$\{x \mapsto \cos(nx) \mid n \in \mathbb{Z}\} \cup \{x \mapsto \sin(nx) \mid n \in \mathbb{Z}\},$$

with the induced inner product from V. Find an orthonormal basis of U consisting of eigenvectors of $\Delta|_U$ and the multiplicities of all eigenvalues.

Lösung:

(a) Partial integration yields

$$\langle Df, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{df}{dx} g\, dx = \frac{1}{2\pi} fg\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} f\frac{dg}{dx}\, dx = -\langle f, Dg\rangle$$

for all f, g, hence $D^* = -D \neq D$. Therefore D is not self adjoint.

(b) Assertion (a) and exercise 6 (a) of Serie 22 yield

$$\Delta^* \;=\; -D^* \circ D^* \;=\; -(-D)\circ(-D) \;=\; \Delta,$$

and hence Δ is self adjoint.

(c) For every $n \neq 0$ set $c_n(x) := \sqrt{2}\cos(nx)$ and $s_n(x) := \sqrt{2}\sin(nx)$. Moreover set $c_0(x) := 1$ and $s_0(x) := 0$.

Claim: The set

$$B := \{c_n \mid n \geqslant 0\} \cup \{s_n \mid n \geqslant 1\}$$

is an orthonormal basis of U consisting of eigenvectors of Δ.

Proof: Since $\sin(-nx) = -\sin(nx)$ and $\cos(-nx) = \cos(nx)$ hold for all $n \geqslant 1$ and $s_0 = 0$, we have that B generates U. The usual derivation rules for trigometric functions imply that for all $n \geqslant 0$, we have

$$\Delta s_n = n^2 s_n \quad \text{and} \quad \Delta c_n = n^2 c_n\,.$$

Hence all elements of B are eigenvectors of Δ.

Next, we show that these are pairwise orthogonal. To see this, we compute for all $n, m \geqslant 0$ using that Δ ist self adjoint:

$$n^2\langle s_n, s_m\rangle = \langle n^2 s_n, s_m\rangle = \langle \Delta s_n, s_m\rangle \overset{!}{=} \langle s_n, \Delta s_m\rangle = \langle s_n, m^2 s_m\rangle = m^2\langle s_n, s_m\rangle.$$

For $n \neq m$ this yields $\langle s_n, s_m\rangle = 0$. Analogous computations show $\langle s_n, c_m\rangle = \langle c_n, s_m\rangle = \langle c_n, c_m\rangle = 0$ for all $n \neq m$. (All of this can also be proved by direct computation of the integrals.) Moreover, we can for all $n \geqslant 1$ insert the formulas

$$\begin{aligned}
\cos^2(nx) &= \frac{\cos(2nx)+1}{2} \\
\sin^2(nx) &= 1-\cos^2(nx) = \frac{1-\cos^2(2nx)}{2} \\
\sin(nx)\cos(nx) &= \frac{1}{2}\sin(2nx),
\end{aligned}$$

into the relevant integrals and compute

$$\langle s_n, s_n \rangle = 1, \quad \langle s_n, c_n \rangle = 0, \quad \langle c_n, c_n \rangle = 1\,.$$

Even simpler, we compute

$$\langle c_0, c_0 \rangle = 1, \quad \langle c_0, s_n \rangle = 0, \quad \langle c_0, c_n \rangle = 0$$

for all $n \geqslant 1$. Together this shows that B is an orthonormal system. As ist generates U it is an orthonormal basis of U.

Together, we see that B is an orthonormal basis of U consisting of eigenvectors of Δ. Every eigenspace of $\Delta|_U$ thus is generated by vectors of B. We conclude that $\Delta|_U$ has eigenvalues

$$0, 1, 4, 9, 16, \ldots, n^2, \ldots$$

with respective multiplicities $1, 2, 2, 2, \ldots$.

Multiple Choice Fragen

1. Let A and B be complex self-adjoint $n \times n$ matrices, and let $\lambda \in \mathbb{C}$. Which of the following statements hold?

   $\boxed{\text{(a)}}$ $A + B$ is self-adjoint.

   (b) λA is self-adjoint.

   $\boxed{\text{(c)}}$ λA is normal.

   Explanation:

   (a) $(A+B)^* = A^* + B^* = A + B$.

   (b) With $\lambda = i$ we have that $(iA)^* = -iA^* = -iA$ is not self adjoint.

   (c) Let $B = \lambda A$. Then $B^*B = \overline{\lambda}A^*\lambda A = |\lambda|^2 AA = BB^*$.

2. Let A, B be complex self-adjoint $n \times n$ matrices and let $\lambda \in \mathbb{C}$. Which of the following statements hold?

   (a) AB is self-adjoint.

   $\boxed{\text{(b)}}$ $AB + BA$ is self-adjoint.

   $\boxed{\text{(c)}}$ $AB - BA$ is normal.

   $\boxed{\text{(d)}}$ ABA is self-adjoint.

   Explanation:

   (a) $(AB)^* = B^*A^* = BA$. Ein Gegenbeispiel ist: $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

   (b) $(AB + BA)^* = B^*A^* + A^*B^* = BA + AB$.

   (c) $(AB - BA)^* = B^*A^* - A^*B^* = -(AB - BA)$. Also:

   $$(AB - BA)^* (AB - BA) = (AB - BA)(AB - BA)^* = -(AB - BA)^2 .$$

   (d) $(ABA)^* = A^*B^*A^* = ABA$.

3. Let A be a normal matrix and $p \in \mathbb{C}[t]$ be a polynomial. Which of the following statements hold?

   (a) $p(A)^* = p(A^*)$.

   $\boxed{\text{(b)}}$ $A^i(A^*)^j = (A^*)^j A^i$.

   $\boxed{\text{(c)}}$ $p(A)$ is normal.

(d) Every eigenvalue λ of A is also an eigenvalue of $p(A)$.

(e) Every eigenvector v of A is also an eigenvector of $p(A)$.

Explanation:

(a) Wrong. $(\lambda A)^* = \overline{\lambda} A^*$. If $p(t) = \sum a_i t^i$, we have $p(A)^* = \sum \overline{a_i} (A^*)^i$.

(b) True. We have $AA^* = A^*A$. Induction yields:

$$A^i (A^*)^j = A^{i-1} A^* A (A^*)^{j-1} = \ldots = A^{i-1} (A^*)^j A = \ldots = (A^*)^j A^i.$$

(c) True. Die $a_i A^i$ $(i \geqslant 0)$ are normal and hence also their sum.

(d) Wrong. A counterexample is $A = 0$, $p(t) = 1$, then $p(A) = 1$, which does not have 0 as eigenvalues.

(e) True. $p(A)v = \sum a_i A^i v = \sum a_i \lambda^i v = p(\lambda) v$.

Note. Let (v_i) be an orthonormal basis of eigenvectors corresponding to the eigenvalues λ_i of A, then (v_i) is an orthonormalbasis of eigenvectors to the eigenvalues $p(\lambda_i)$ of $p(A)$.