Musterlösung Serie 23

Positive-definiteness, isometries

1. Let K be a field in which $2 \neq 0$, V a K-vector space, and let B be a symmetric bilinear form on V. We define $q_B(v) = B(v,v)$ for every $v \in V$ to be the quadratic form associated to B. Show that

$$B(v,w) = \frac{1}{2}(q_B(v+w) - q_B(v) - q_B(w)).$$

Solution: This is straight-forward using the bilinearity and the symmetry of B. We have

$$\begin{aligned} q_B(v+w) - q_B(v) - q_B(w) &= B(v+w, v+w) - B(v,v) - B(w,w) \\ &= B(v, v+w) + B(w, v+w) - B(v,v) - B(w,w) \\ &= B(v,w) + B(w,v) \quad \textit{(linearity in both entries)} \\ &= 2B(v,w) \quad \textit{(symmetry)}. \end{aligned}$$

2. Consider the real matrix

$$A := \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ -1 & -2 & -2 \\ 2 & 1 & -2 \end{pmatrix}.$$

(a) Show that A is orthogonal and $\det A = 1$.

(b) Determine the rotational axis and the angle of $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$.

Lösung:

(a) Direct calculations show $A^T A = I_3$ and $\det A = 1$.

(b) By (a) the map T_A is an element of SO_3, thus it is a rotation. The rotational axis is contained in the eigenspace of A corresponding to eigenvalue 1, which is

$$\mathrm{Eig}_A(1) = \left\langle \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\rangle.$$

Moreover, there exists an ordered orthonormal basis $\mathcal{B}$ with

$$[A]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix},$$

for an rotational angle $\varphi \in [-\pi, \pi]$. We get

$$1 + 2\cos\varphi = \mathrm{Tr}([A]_{\mathcal{B}}^{\mathcal{B}}) = \mathrm{Tr}(A) = -\frac{2}{3},$$

and thus

$$\varphi \;=\; \arccos(-5/6) \;=\; \pi - \arccos(5/6) \;\approx\; 146.44°.$$

Aliter: Once you have identified the axis of rotation, find a vector that is orthogonal to it, e.g.

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$$

Now, its image Av, which we denote w, is also in the plane orthogonal to the axis of rotation and we can calculate the angle φ using

$$\varphi = \arccos(\cos(\varphi)) = \arccos\left(\frac{\langle v, w\rangle}{||v||\,||w||}\right).$$

3. Which of the following three real symmetrix matrices are positive definite?

$$A := \begin{pmatrix} 3 & 3 & 2 & 3 \\ 3 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 3 \end{pmatrix}, \quad B := \begin{pmatrix} 6 & 3 & 4 \\ 3 & 7 & 3 \\ 4 & 3 & 8 \end{pmatrix}, \quad C := \begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & 6 & 1 & 1 \\ -1 & 1 & 8 & 2 \\ 0 & 1 & 2 & 5 \end{pmatrix}.$$

Hinweis: Verwende das Hauptminorenkriterium.

Lösung: Let A_i, B_i, C_i be the corresponding first minors.

As $\det A_2 = \det\left(\begin{smallmatrix} 3 & 3 \\ 3 & 1 \end{smallmatrix}\right) = -6$ is negative, the matrix A is not positive definite. However, we have $\det B_i = 6,\ 33,\ 170$ for $i = 1,\ 2,\ 3$ all positive, hence B is positive definite. Similarly, we have $\det C_i = 3,\ 18,\ 135,\ 592$ for $i = 1, \ldots, 4$ all positive, hence C is positive definite as well.

4. Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix. Show that the following statements are equivalent:

   (A) A is positive definite, i.e. $v^T Av > 0$ for all $v \neq 0$;

   (B) All eigenvalues of A are positive;

   (C) There exists an invertible symmetric matrix $S \in M_{n\times n}(\mathbb{R})$ such that $S^2 = A$.

   Solution: (A) $\implies$ (B): Let v be an eigenvector of A associated to an eigenvalue λ. Without loss of generality, we may assume that it is normalised so that $x^T x = 1$. Since we assume (A), we have

$$0 < x^T AX = x^T \lambda x = \lambda x^T x = \lambda.$$

(B) $\implies$ (C): Since A is symmetric, by the spectral theorem there exists a matrix $Q \in \mathrm{O}_3(\mathbb{R})$ such that

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} =: D,$$

where $\{\lambda_i\}$ are the (not necessarily distinct) eigenvalues of A. Let D' be the diagonal matrix with the $\sqrt{\lambda_i}$'s on the diagonal. Then $D'^2 = D = Q^{-1}AQ$, so

$$(QD'Q^{-1})^2 = Q(D')^2Q^{-1} = A.$$

Since $Q \in \mathrm{O}_3$, we have $Q^{-1} = Q^T$. From there you can easily observe that $QD'Q^T$ is symmetric and invertible.

(C) $\implies$ (A): Let S be an invertible symmetric real matrix such that $S^2 = A$. Since S is invertible, we have

$$\forall v \in \mathbb{R}^n : S \cdot v = 0 \iff v = 0.$$

Hence, for any $v \in \mathbb{R}^n \setminus \{0\}$,

$$v^TAv = v^TS^2v = v^TS^TSv = (Sv)^T(Sv) = ||Sv||^2 > 0.$$

5. Show: For every orthogonal endomorphism f of an n-dimensional Euclidean vectorspace V, we have

$$|\operatorname{Tr}(f)| \leqslant n.$$

For which f do we have equality?

Lösung: Note that $\operatorname{Tr}(f) = \operatorname{Tr}(A)$ for any matrix $A \in M_{n\times n}(\mathbb{R})$ representing f with respect to an orthonormal basis $\mathcal{B}$. We show that A is an orthogonal matrix, i.e. $A^TA = I_n = AA^T$.

Indeed, since for any $u, v \in V$ it holds that $\langle fu, fv\rangle = \langle u, v\rangle$, we have

$$(Au)^TM(Av) = u^TA^TMAv = u^TMv,$$

where M is the matrix representing the inner product with respect to $\mathcal{B}$. Since this holds for any pair $u, v \in V$, we have

$$A^TMA = M.$$

It follows from the fact that $\mathcal{B}$ is orthonormal that $M = I_n$. Thereupon, $A^TA = I_n$. We deduce that the columns of A form an orthonormal basis of V, and therefore that each diagonal entry a_{ii} of A has norm smaller than 1. This shows

$$\operatorname{Tr}(f) = \operatorname{Tr}(A) \leqslant n.$$

Finally, equality will hold if every entry on the diagonal has norm exactly 1 and if they all have the same sign, i.e. if $A = I_n$ or $A = -I_n$. In turn, we conclude that f is the identity or minus the identity in this case.

Aliter: By the spectral theorem, there exists an oredered orthonormal basis $\mathcal{B}$ of V, such that the representation matrix of f with respect to $\mathcal{B}$ is of the form

$$[M_f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_r \end{pmatrix},$$

where D_k is ± 1 or $\left(\begin{smallmatrix} a_k & b_k \\ -b_k & a_k \end{smallmatrix}\right)$ with $a_k^2 + b_k^2 = 1$.

In the first case, we have $|\operatorname{Tr}(D_k)| = 1$ and in the second case $|\operatorname{Tr}(D_k)| = |2a_k| \leqslant 2$. Together, this yields

$$|\operatorname{Tr}(f)| = |\sum_{k=1}^{r} \operatorname{Tr}(D_k)| \leqslant \sum_{k=1}^{r} |\operatorname{Tr}(D_k)| \leqslant n.$$

Equality holds if and only if for all D_k of the second type $|\operatorname{Tr}(D_k)| = 2$ holds and if all diagonal entries of $[M_f]_{\mathcal{B}}^{\mathcal{B}}$ have the same sign. This is the case if and only if $f = \pm \operatorname{id}_V$.

Aliter w/o spectral theorem: Let $\mathcal{B} = (b_1, \ldots, b_n)$ be an arbitray ordered orthomormal basis of V. For every $1 \leqslant j \leqslant n$ we have

$$f(b_j) = \sum_{i=1}^{n} \langle b_i, f(b_j) \rangle \cdot b_i.$$

Thus f has a representation matrix $[M_f]_{\mathcal{B}}^{\mathcal{B}} = (\langle b_i, f(b_j) \rangle)_{ij}$. This yields

$$\operatorname{Tr}(f) = \operatorname{Tr}([M_f]_{\mathcal{B}}^{\mathcal{B}}) = \sum_{i=1}^{n} \langle b_i, f(b_i) \rangle.$$

As f is orthogonal, for every i we have $\|b_i\| = \|f(b_i)\| = 1$. The Cauchy-Schwarz-inequality yields

$$|\langle b_i, f(b_i) \rangle| \leqslant \sqrt{||b_i||^2 \, ||f(b_j)||^2} = 1.$$

Summation gives $|\operatorname{Tr}(f)| \leqslant n$, as desired.

Moreover, we have $|\operatorname{Tr}(f)| = n$ if and only if the real numbers $\langle b_i, f(b_i) \rangle$ are all 1 or all -1. In this case we have for every i equality in the Cauchy-Schwarz-inequality, and hence $f(b_i)$ is linear dependent of b_i. Thus $f(b_i) = b_i$ for all i, or $f(b_i) = -b_i$ for all i, therefore $f = \pm \operatorname{id}_V$.

6. Consider two 2-dimensional subspaces $E_1, E_2 \subset \mathbb{R}^3$. Describe the set of elements $T \in \mathrm{SO}_3(\mathbb{R})$ such that

$$TE_1 = E_2,$$

in terms of orthogonal bases of E_1 and E_2.

Hint: Start by assuming that $E_1 = E_2 = \mathrm{Sp}(e_1, e_2)$.

Solution: We start by determining and fixing two orthonormal bases v_1, v_2, v_3 and w_1, w_2, w_3 of $\mathbb{R}$ with $v_1, v_2 \in E_1$ and $w_1, w_2 \in E_2$. This yields orthogonal matrices $M_1 := (v_1 v_2 v_3)$ and $M_2 := (w_1 w_2 w_3)$. These have determinant ± 1 and after replacing v_3 by $-v_3$ or w_r with $-w_3$ respectively if necessary, we can reach dass $\det(M_1) = \det(M_2) = 1$.

Note that rotations of $\mathbb{R}^3$ form the Group SO(3) of all orthogonal matrices with determinant 1. In particular, every composition of rotations and the inverses of every rotations is again a rotation, and by construction M_1 and M_2 are rotations.

Now let E_0 be the subvectorspace spanned by the standard basis vectors e_1, e_2. By construction we have $M_1 E_0 = E_1$ and $M_2 E_0 = E_2$. For an arbitrary rotation T we then have

$$TE_1 = E_2 \Longleftrightarrow TM_1E_0 = M_2E_0 \quad \Longleftrightarrow \quad M_2^{-1}TM_1E_0 = E_0.$$

Hence $D := M_2^{-1}TM_1$ is a rotatoin with $DE_0 = E_0$, and $T = M_2DM_1^{-1}$. It remains to determine all rotations D with $DE_0 = E_0$. Such a rotation needs to map the orthogonal complement of E_0 onto itself. This is generated by e_3. By $\|De_3\| = \|e_3\| = 1$ we then have $De_3 = \pm e_3$.

In the case of $De_3 = e_3$, we have that D is a rotation around the axis $\mathbb{R}e_3$, and all of these are given by the matrices

$$D_\varphi := \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $(a, b) = (\cos\varphi, \sin\varphi)$ with $\varphi \in \mathbb{R}$. Then

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is a rotation with $SE_0 = E_0$ and $Se_3 = -e_3$. In the case $De_3 = -e_3$ we thus have that $S^{-1}D$ is a rotation with $S^{-1}DE_0 = E_0$ and $S^{-1}De_3 = e_3$. Therefore $S^{-1}D = D_\varphi$ for a φ and thus $D = SD_\varphi$. The set of all rotations D with $DE_0 = E_0$ therefore is

$$\{D_\varphi \mid \varphi \in \mathbb{R}\} \cup \{SD_\varphi \mid \varphi \in \mathbb{R}\}$$

The set of all rotations T with $TE_1 = E_2$ hence is

$$\left\{M_2 D_\varphi M_1^{-1} \mid \varphi \in \mathbb{R}\right\} \cup \left\{M_2 S D_\varphi M_1^{-1} \mid \varphi \in \mathbb{R}\right\}.$$