Musterlösung Serie 24

Bilinear forms, Singular values decomposition, Jordan Normal form

1. (a) Determine a singular value decomposition $A = QDR$ of the real matrix

$$A := \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) Determine a singular value decomposition of A^T.

Lösung:

(a) We compute the matrix

$$A^T A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the corresponding characteristic polynomial

$$P_{A^T A}(X) = \det \begin{pmatrix} 2-X & -1 \\ -1 & 2-X \end{pmatrix} = X^2 - 4X + 3 = (X-3)(X-1).$$

Hence $A^T A$ has Eigenvalues $\lambda_1 := 3$ and $\lambda_2 := 1$. The singular values of A therefore are $\sigma_1 := \sqrt{3}$ and $\sigma_2 := 1$ and the matrix D is

$$D := \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The normed eigenvectors of $A^T A$ corresponding to the eigenvalues λ_1 resp. λ_2 are

$$v_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{resp.} \quad v_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These are the columns of R^T, thus we consider the orthogonal Matrix

$$R := (v_1 \ v_2)^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The desired equation $AR^{-1} = QD$ implies that for $i = 1, 2$ the i-th column of Q is equal to $1/\sigma_i$ times the i-th column of AR^{-1}. Thus we have $Q := (w_1\ w_2\ w_3)$ with

$$w_1 := \frac{1}{\sigma_2} A v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 := \frac{1}{\sigma_1} A v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and an arbitrary vector w_3, such that (w_1, w_2, w_3) is an orthonormal basis. For example, Gram-Schmidt for the basis (w_1, w_2, e_1) with $e_1 := (1, 0, 0)^T$ yields

$$w_3 := \frac{e_1 - \langle e_1, w_1 \rangle w_1 - \langle e_1, w_2 \rangle w_2}{||e_1 - \langle e_1, w_1 \rangle w_1 - \langle e_1, w_2 \rangle w_2||} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

The desired decomposition thus is

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = A = QDR = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ 1 & \sqrt{3} & -\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

(b) The transpose of an orthogonal matrix is again orthogonal. Thus $A^T = R^T D^T Q^T$ is again a singular value decomposition.

2. Consider the real matrix

$$A = \begin{pmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

Find a matrix $P \in \mathrm{O}_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Solution: ONB:

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

consisting of eigenvectors of A such that

$$P^{-1}AP = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -15 \end{pmatrix}.$$

3. Let V be an n-dimensional vector space over a field K. Show the following statements using the lemma about generalized eigenspaces seen in the lectures, but without using the Jordan Normal Form theorem:

(a) Suppose that $N \in \operatorname{End}(V)$ is nilpotent. Then 0 is an eigenvalue of N and it is the only one.

(b) If $N \in \operatorname{End}(V)$ is nilpotent, then $N^n = O_{n\times n}$. In other words, the nilpotency index of N is smaller or equal to $\dim(V)$.

(c) Suppose that $N \in \operatorname{End}(V)$ is nilpotent and assume that $p_N(x)$ splits as a product of linear factors in $K[x]$. Then $p_N(x) = (-x)^n$.

(d) Let $T \in \operatorname{End}(V)$ and assume that $p_T(x)$ splits into linear factors in $K[x]$. Let $\eta \in K$ and define $S = T - \eta \operatorname{Id}_V$. Then $p_S(x)$ also splits into a product of linear factors over $K[x]$. In fact, $p_S(x) = p_T(x + \eta)$.

(e) Let $T \in \operatorname{End}(V)$ and assume that $\lambda \in K$ is the only eigenvalue of T and that $p_T(x)$ splits as a product of linear factors in $K[x]$. Define $N = T - \lambda \operatorname{Id}_V$. Then $p_N(x) = (-x)^n$, and $N^n = O_{n\times n}$.

Solution:

(a) This was shown in serie 16, exercise 5.

(b) By (a), 0 is an eigenvalue of N. We consider the generalised eigenspace $\widetilde{\operatorname{Eig}}_N(0)$. We have

$$\widetilde{\operatorname{Eig}}_N(0) = \bigcup_{k=0}^{\infty} \ker(N^k) = V,$$

since there exists an integer m such that N^m vanishes. By lemma 1 in the notes jordan.b, we also have

$$\widetilde{\operatorname{Eig}}_N(0) = \ker(N^n).$$

Hence $V = \ker(N^n)$, which implies that N^n vanishes.

(c) Since $p_N(x)$ splits as a product of linear factors in $K[x]$, we can write it as

$$p_N(x) = \prod_{k=1}^{n} (\lambda_i - x),$$

where the λ_k's are the (not necessarily distinct) eigenvalues of N. It follows from (a) that $p_N(x) = (-x)^n$.

(d) For $\mu \in K$, we have

$$(T - \eta \operatorname{Id}_V)u = \mu u \quad \Longleftrightarrow \quad Tu = (\eta + \mu)u.$$

Thereupon, a scalar $\mu \in K$ is an eigenvalue of S if and only if $\mu + \eta$ is an eigenvalue of T. Since $p_T(x)$ splits into linear factors in $K[x]$, we have

$$\begin{aligned} p_T(x+\eta) &= \prod_{\lambda \text{ eigenvalue of } T} (\lambda - (x+\eta)) \\ &= \prod_{\lambda \text{ eigenvalue of } T} (\lambda - \eta - x) \\ &= \prod_{\lambda \text{ eigenvalue of } S} (\mu - x) \\ &= p_S(x). \end{aligned}$$

Note that $p_S(x)$ splits into a product of linear factors since its roots are the set $\{\lambda - \eta \mid \lambda \text{ is an eigenvalue of } T\}$, and by assumption, the eigenvalues of T are in K and so is η.

(e) We start by showing that N is nilpotent. Indeed, by the first , proposition on page 6 of the lecture notes jordan.b, we have

$$\begin{aligned} V = \oplus_{\eta \text{ eigenvalue of } T} \tilde{\mathrm{Eig}}_T(\eta) &= \tilde{\mathrm{Eig}}_T(\lambda) \\ &= \bigcup_{k=1}^{\infty} \ker((T - \lambda \, \mathrm{Id}_V)^k) \\ &= \ker((T - \lambda \, \mathrm{Id}_V)^n), \end{aligned}$$

where we used lemma 1 in the same lecture notes to obtain the last equality. Hence N^n is the trivial endomorphism. Now, it follows from (c) that $p_N(x) = (-x)^n$.

4. Determine the Jordan normal form of the following matrix over $\mathbb{R}$ and over $\mathbb{F}_3$:

$$A := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Lösung: Over $\mathbb{R}$, the characteristic polynomial of A is $(X-1)^2(X-4)$. The eigenspace to the eigenvalue 4 thus has dimension 1. Next, we compute rank$(A - I_3)$ $= 1$; hence the eigenspace to eigenvalue 1 has dimension 2. Therefore, there eixists a basis consisting of eigenvectors and the matrix is diagonalisable over $\mathbb{R}$ with Jordan normal form

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Over $\mathbb{F}_3$ the characteristic polynomial is equal to $(X-1)^3$; thus A has exactly one eigenspace, which corresponds to $X - 1$. We compute

$$A - I_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and $(A-I_3)^k = 0$ für $k \geqslant 2$. From $\dim \operatorname{Kern}(A-I_3) = 2$ it follows that there exist Jordan blocks of size 1 and 2. Thus the matrix A over $\mathbb{F}_3$ has Jordan normal form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. Determine the Jordan normal form and the corresponding base change matrices of the real matrix

$$A := \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Lösung: The characteristic polynomial of A is

$$\operatorname{char}_A(X) \;=\; X^4 - 11X^3 + 45X^2 - 81X + 54 \;=\; (X-2)\cdot(X-3)^3\,.$$

We treat 2 and 3 separately.

Eigenvalue 2: The space $\widetilde{\operatorname{Eig}}_A(2)$ is one-dimensional and equal to the eigenspace of A corresponding to 2. We compute $\operatorname{Ker}(A - 2\operatorname{Id}_4)$ and find the eigenvector

$$v_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Eigenvalue 3: For $B := A - 3\operatorname{Id}_4$, we have

$$B = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B^2 = \begin{pmatrix} 1 & -2 & -2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -B^3.$$

This yields

k	1	2	3	4	$\cdots$
$\operatorname{rank}(B^k)$	2	1	1	1	$\cdots$
$\dim \operatorname{Ker}(L_{B^k})$	2	3	3	3	$\cdots$
# $k \times k$-Jordan block to EV 3	1	1	0	0	$\cdots$

We compute

$$\widetilde{\mathrm{Eig}}_A(3) \;=\; \mathrm{Ker}(L_{B^3}) \;=\; \left\langle \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-6\\0\\0\\1\end{pmatrix} \right\rangle .$$

Next, we search for a vector $v_2 \in \widetilde{\mathrm{Eig}}_A(3)$, whose image under L_B is non-zero. One example is

$$v_2 := \begin{pmatrix}-6\\0\\0\\1\end{pmatrix} \quad \text{with} \quad Bv_2 = \begin{pmatrix}8\\2\\2\\0\end{pmatrix}.$$

We search for another vector $v_3 \in \mathrm{Kern}(L_B) \setminus \langle Bv_2 \rangle$, e.g.

$$v_3 := \begin{pmatrix}2\\1\\0\\0\end{pmatrix}.$$

Then v_2, Bv_2, v_3 is a basis of $\widetilde{\mathrm{Eig}}_A(3)$.

Combining the cases: By the decomposition into generalized eigenspaces $b := (v_1, Bv_2, v_2, v_3)$ is a basis $\mathbb{R}^4$. By construction, we have $Av_1 = 2v_1$ and $A(Bv_2) = 3(Bv_2)$ and $Av_2 = Bv_2 + 3v_2$ as well as $Av_3 = 3v_3$. For the base change matrix, we have

$$S \;:=\; (v_1 \mid v_3 \mid Bv_2 \mid v_2) \;=\; \begin{pmatrix}1 & 2 & 8 & -6\\ 0 & 1 & 2 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

hence

$$S^{-1}AS \;=\; \left(\begin{array}{c|c|cc} 2 & 0 & 0 & 0\\ \hline 0 & 3 & 0 & 0\\ \hline 0 & 0 & 3 & 1\\ 0 & 0 & 0 & 3\end{array}\right).$$

This is the Jordan normal form of A.

6. *Example regarding special relativity.* Define the symmetric blinear form$s \colon \mathbb{R}^4 \to \mathbb{R}^4$ for all $v = (x, y, z, t)^T$ and $v' = (x', y', z', t')^T$ in $\mathbb{R}^4$ by

$$s(v, v') := xx' + yy' + zz' - ctt',$$

where $c > 0$ is a fixed parameter. The space $M := (\mathbb{R}^4, s)$ is called *Minkowski space* (sometimes Minkowski spacetime) and the parameter c is called *light speed.* We use the normalization $c = 1$.

A linear map $F : M \to M$ is called *isometry* or *Lorentz transformation*, if

$$\forall v, w \in \mathbb{R}^4 : \; s\big(F(v), F(w)\big) = s(v, w) \,.$$

(a) Show that every isometry is bijective.

(b) Show that the following endomorphisms are isometries of M:

  i. Left multiplication with $\left(\begin{array}{c|c} T & 0 \\ \hline 0 & \pm 1 \end{array}\right)$ für jedes $T \in O(3)$.

  ii. A *Lorentz boost* in x-direction with speed $v < c = 1$, given by left multiplication with the matrix

$$B := \begin{pmatrix} \gamma & & & -v\gamma \\ & 1 & & \\ & & 1 & \\ -v\gamma & & & \gamma, \end{pmatrix}$$

  for $\gamma := 1/\sqrt{1 - v^2}$.

(c) The subset $\{x \in M \mid s(x, x) = 0\}$ is called *light cone in M*. Prove the „relativistic football theorem“: Every linear isometry φ with $\det(\varphi) = 1$ has an eigenvector in the light cone.

Remark. For $c \to \infty$ the light cone approaches the subspace $\{t = 0\}$ and the statement reduces to the classical case.

Lösung:

(a) Let $F\colon M \to M$ be an isometry, and let $v = (x_1, \ldots, x_4)^T$ be an arbitrary element contained in the kernel of F. Denote by $e_1, \ldots, e_4$ the standard basis of $M = \mathbb{R}^4$. Then for all $i = 1, \ldots, 4$ we have

$$\pm\, x_i \;=\; s(v, e_i) \;=\; s(F(v), F(e_i)) \;=\; s(0, e_i) \;=\; 0,$$

and so $v = 0$. This means that $\mathrm{Kern}(F) = \{0\}$, and as injective endomorphism of finite-dimensional vector spaces F is also bijective.

(b) (i) Follows from direct computations

  (ii) For all $v = (x, y, z, t)^T$ and $v' = (x', y', z', t')^T$ in M we have

$$\begin{aligned} s(Bv, Bv') &= (\gamma x - v\gamma t)(\gamma x' - v\gamma t') + yy' + zz' - (-v\gamma x + \gamma t)(-v\gamma x' + \gamma t') \\ &= (\gamma^2 - v^2\gamma^2)xx' + yy' + zz' - (\gamma^2 - v^2\gamma^2)tt' \\ &= xx' + yy' + zz' - tt' \\ &= s(v, w)\,, \end{aligned}$$

  hence the Lorentzboost L_B is an isometry.

(c) Let $\varphi : M \to M$ be a linear isometry with $\det(\varphi) = 1$.

Step 1: There exists an φ-invariant subspace U of dimension 2.

Proof: Every irreducible factor of the characteristic polynomial of φ had degree 1 or 2. If there exists an irreducible factor of degree 2 we can write the Jordan normal form of φ with an 2×2-block in the upper left corner. Otherweise, all irreducible factors have degree 1 and the Jordan normal form of φ is an upper triangular matrix. In both cases, the first two basis vectors generate an φ-invariant subspace of dimension 2. □

Step 2: The „orthogonal complement“

$$U^{\perp} = \{v \in M \mid \forall u \in U \colon s(u,v) = 0\}$$

also is a φ-invariant subspace of dimension 2.

Proof: As in the case of a scalar product, since s is non-degenerate. □

Step 3: We have $U \subsetneq U^{\perp}$.

Proof: The restriction of s on the subspace V generated by $t = 0$ is positive definite. Moreover, we have

$$\begin{aligned} \dim(U \cap V) &= \dim(U) + \dim(V) - \dim(U+V) \\ &\geqslant \dim(U) + \dim(V) - \dim(M) \;=\; 2+3-4 \;=1. \end{aligned}$$

Hence, there exists a non-zero vector $u \in U \cap V$, for which we have $s(u,u) > 0$. Hence $u \notin U^{\perp}$.

Step 4: Proof in the case of $U \cap U^{\perp} \neq 0$.

Proof: Using steps 1 and 2 we get that $U \cap U^{\perp}$ is an φ-invariant subspaces, and by step 3 it has dimension 1. Every non-zero vector u contained there hence is an eigenvector of φ. By definition of $U^{\perp}$ it satisfies $s(u,u) = 0$, as desired. □

From now on, we assume $U \cap U^{\perp} = 0$. Then we have $M = U \oplus U^{\perp}$.

Step 5: After exchanging U and $U^{\perp}$ if possible we can assume that s is positive definite on U and indefinite on $U^{\perp}$.

Proof: Consider any ordered basis of U and extend it with an ordered basis of $U^{\perp}$ to a basis B of M. By definition of $U^{\perp}$ the representation matrix $[s]_B$ is a block diagonal matrix with blocks of size 2. The signature of s thus is the sum of the signatures of $s|_{U \times U}$ and $s|_{U^{\perp} \times U^{\perp}}$. As s has signature $(3,1)$ one of the restrictions needs to habe signature $(2,0)$ and the other signature $(1,1)$. □

Step 6: For the restrictions of the given isometry

$$\varphi_U := \varphi|_U : U \to U \qquad \text{and} \qquad \varphi_{U^\perp} := \varphi|_{U^\perp} : U^\perp \to U^\perp$$

we have $\det(\varphi_U) = \det(\varphi_{U^\perp}) = \pm 1$.

Proof: The restriction φ_U is an isometry with respect to $s|_{U\times U}$, hence we have $\det(\varphi_U) = \pm 1$. Moreover, by assumption we get

$$\det(\varphi_U) \cdot \det(\varphi_{U^\perp}) \;=\; \det(\varphi) \;=\; 1$$

Together this yields $\det(\varphi_{U^\perp}) = \det(\varphi_U)$. □

Step 7: Proof in the case $\det(\varphi_U) = \det(\varphi_{U^\perp}) = -1$.

Proof: Here φ_U is an isometry of the 2-dimensional Euclidean vectorspace U with determinant -1, hence it is a reflection with eigenvalues $+1$ and -1. Moreover, we have that $\varphi_{U^\perp}$ is an endomorphism of the 2-dimensional real vectorspace $U^\perp$ with determinant -1. Hence the charakteristisc polynomial splits into linear factors over $\mathbb{R}$ and thus there exists an eigenvector $v \in U^\perp$, we call its eigenvalue λ. If $s(v,v) = 0$, this is the desired eigenvector in the light cone. Otherwise, the computation

$$s(v,v) \;=\; s(\varphi(v),\varphi(v)) \;=\; s(\lambda v, \lambda v) \;=\; \lambda^2 \cdot s(v,v),$$

yields $\lambda = \pm 1$. As $\det(\varphi_{U^\perp}) = -1$, we get that $\varphi_{U^\perp}$ also needs to have eigenvalue $-1/\lambda = \mp 1$. Together this shwos that φ has eigenvalues ± 1 with respective multiplicity 2.

Both Eigenspaces are then φ-invariant subspaces of dimension 2. After replacing U and $U^\perp$ by these, the restriction $\varphi_{U^\perp}$ thus is scalar. As $s|_{U^\perp\times U^\perp}$ is indefinite, there exists a vector $v \in U^\perp$ with $s(v,v) = 0$. This is the searched eigenvector in the light cone. □

Step 8: Proof in the case of $\det(\varphi_U) = \det(\varphi_{U^\perp}) = 1$.
Proof: As $s|_{U^\perp\times U^\perp}$ is indefinite, there exists a basis B of $U^\perp$ with

$$[s|_{U^\perp\times U^\perp}]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the representation matrix $A := {}_B[\varphi_{U^\perp}]_B$ we have $A^T\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\det(A) = 1$. Direct computations show that these conditions are equivalent to

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

for $a, b \in \mathbb{R}$ with $a^2 - b^2 = 1$. The column vector $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ corresponds to an eigenvector $\varphi_{U^\perp}$ with eigenvalue $a + b$ in the light cone. □

Single Choice. In each exercise, exactly one answer is correct.

1. Let f be an endomorphism of a finite-dimensional vector space V and let λ be an eigenvalue of f. Which statement is generally false?

   (a) Every eigenvector of f corresponding to the eigenvalue λ lies in the eigenspace $\widetilde{\mathrm{Eig}}_f(\lambda)$.

   [(b)] Every vector in $\widetilde{\mathrm{Eig}}_f(\lambda)$ is an eigenvector of f corresponding to the eigenvalue λ.

   (c) The generalized eigenspace $\widetilde{\mathrm{Eig}}_f(\lambda)$ is not the zero space.

   (d) For every eigenvalue μ of f with $\mu \neq \lambda$, we have $\widetilde{\mathrm{Eig}}_f(\mu) \cap \widetilde{\mathrm{Eig}}_f(\lambda) = \langle 0 \rangle$.

   Explanation: The generalized eigenspace of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with respect to $X - 1$ is two-dimensional, but the eigenspace is one-dimensional.

2. For every endomorphism f of an n-dimensional vector space V whose characteristic polynomial factors into linear factors, and every eigenvalue λ of f we have:

   (a) $\widetilde{\mathrm{Eig}}_f(\lambda) = \mathrm{Kern}(f - \lambda\,\mathrm{id}_V)$.

   (b) $\dim(\widetilde{\mathrm{Eig}}_f(\lambda)) = 1$.

   (c) $\dim(\widetilde{\mathrm{Eig}}_f(\lambda)) = n$.

   [(d)] $\widetilde{\mathrm{Eig}}_f(\lambda) = \mathrm{Kern}((f - \lambda\,\mathrm{id}_V)^n)$.

   Explanation: Let m be the algebraic multiplicity of λ. Then we have $m \leqslant n$ and $\widetilde{\mathrm{Eig}}_f(\lambda) = \mathrm{Kern}((f - \lambda\,\mathrm{id}_V)^m) \subset \mathrm{Kern}((f - \lambda\,\mathrm{id}_V)^n)$. By the decomposition formula with respect to generalized eigenspaces, it follows that this inclusion is an equality.

3. The generalized eigenspace of the real matrix $A := \begin{pmatrix} 2 & 3 & -1 & 5 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ with respect to $X - 2$ is

   (a) One-dimensional

   (b) Two-dimensional

   [(c)] Three-dimensional

   (d) Four-dimensional

   Explanation: Regardless of what is above the diagonal, the characteristic polynomial is $(X-2)^3(X-3)$; therefore, the generalized eigenspace with respect to $X-2$ has dimension 3.

4. Let A be a 3×3 matrix with $A \neq 0$ and $A^2 = 0$. Then, the Jordan normal form of A has

   (a) 1 Jordan block.

   [(b)] 2 Jordan blocks.

   (c) 3 Jordan blocks.

   (d) It depends on the exact matrix A.

   Explanation: A nilpotent matrix has only the eigenvalue 0, so the possible Jordan normal forms are $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. The first matrix has a non-zero square, and because A is also not the zero matrix, only the middle option remains, so the Jordan normal form has two Jordan blocks.

Multiple Choice Fragen

1. Which of the following statements is **true**: For arbitrary integers $n > m \geqslant 1$, there exists a square matrix with...

   (a) characteristic polynomial $X^m + X^n$.

   (b) minimal polynomial X^m and characteristic polynomial X^n.

   (c) minimal polynomial $X^m \cdot (X^n - 1)$.

   Explanation: The companion matrix of a polynomial has exactly this polynomial as its minimal and characteristic polynomial; hence, (a) and (c) are correct. Also, (b) is correct, for example, by taking a block diagonal matrix with a Jordan block of size m and $n - m$ Jordan blocks of size 1.