Musterlösung Serie 25

Jordan Normal form, Multilinear algebra

1. Prove the following propositions:

   (a) For all K-vector spaces $V_1, \ldots, V_r$ and W, we have that $\mathrm{Mult}_K(V_1, \ldots, V_r; W)$ is a subspace of the vector space of all maps $V_1 \times \cdots \times V_r \to W$.

   (b) Consider linear maps of K-vector spaces $f_i \colon V_i' \to V_i$ for $1 \leqslant i \leqslant r$ as well as $g \colon W \to W'$. Then we get a linear map

$$\begin{aligned} \mathrm{Mult}_K(V_1, \ldots, V_r; W) &\to \mathrm{Mult}_K(V_1', \ldots, V_r'; W'), \\ \varphi &\mapsto g \circ \varphi \circ (f_1 \times \cdots \times f_r). \end{aligned}$$

Lösung:

(a) For every $1 \leqslant i \leqslant r$ and $v_j \in V_j$ for $j \neq i$ consider the map

$$\varepsilon: \ V_i \longrightarrow V_1 \times \cdots \times V_r, \quad v \mapsto (v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_r).$$

We note that ε depends on the choices of $i, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r$, but suppress this in the notation in orter to aid readability. By definition, the map $\varphi \colon V_1 \times \cdots \times V_r \to W$ is multilinear if and only if for every choice of $i, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r$ the composition $\varphi \circ \varepsilon \colon V_i \to W$ is linear.

For the zero map $\varphi_0 \colon V_1 \times \cdots \times V_r \to W$ is $\varphi_0 \circ \varepsilon$ again the zero map, hence linear. Now let $\varphi_1, \varphi_2 \colon V_1 \times \cdots \times V_r \to W$ be multilinear maps and let $\lambda \in K$. Then $\varphi_1 \circ \varepsilon$ and $\varphi_2 \circ \varepsilon$ are linear and thus $(\varphi_1 + \varphi_2) \circ \varepsilon_i = \varphi_1 \circ \varepsilon_i + \varphi_2 \circ \varepsilon_i$ and $(\lambda \cdot \varphi_1) \circ \varepsilon = \lambda \cdot (\varphi_1 \circ \varepsilon)$ are linear as well, because we know that linear combinations of linear maps are linear. Varration of $i, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r$ and so of ε yields that φ_0 and $\varphi_1 + \varphi_2$ and $\lambda \cdot \varphi_1$ are multilinear.

Together, this shows that $\mathrm{Mult}_K(V_1, \ldots, V_r; W)$ is a subspace.

(b) For every $1 \leqslant i \leqslant r$ and $v_j' \in V_j'$ for $j \neq i$ consider the map

$$\varepsilon': \ V_i' \longrightarrow V_1' \times \cdots \times V_r', \quad v' \mapsto (v_1', \ldots, v_{i-1}', v', v_{i+1}', \ldots, v_r').$$

By definition, a map $\varphi' \colon V_1' \times \cdots \times V_r' \to W'$ is multilinear if and only if for all $i, v_1', \ldots, v_{i-1}', v_{i+1}', \ldots, v_r'$ the composite map $\varphi' \circ \varepsilon' \colon V_i' \to W'$ is linear. Moreover, set $v_j := f_j(v_j')$ for all $j \neq i$ and $\varepsilon \colon V_i \to V_1 \times \cdots \times V_r$ like in (a). Then we have $(f_1 \times \cdots \times f_r) \circ \varepsilon' = \varepsilon \circ f_i$.

Consider a linear map $\varphi \in \mathrm{Mult}_K(V_1, \dots, V_r; W)$. For every choice of $i, v'_1, \dots, v'_{i-1}, v'_{i+1}, \dots, v'_r$ and thus of ε' and ε as above, we have that $\varphi \circ \varepsilon \colon V_i \to W$ is linear. Then

$$g \circ \varphi \circ (f_1 \times \cdots \times f_r) \circ \varepsilon' = g \circ (\varphi \circ \varepsilon) \circ f_i$$

is also linear as composition of linear maps. Hence $g \circ \varphi \circ (f_1 \times \cdots \times f_r)$ is multilinear; the map from the proposition is thus well defined.
Now consider multilinear maps $\varphi_1, \varphi_2 \colon V_1 \times \cdots \times V_r \to W$ and $\lambda \in K$. Since g is linear, we have

$$g \circ (\lambda\varphi_1 + \varphi_2) \circ (f_1 \times \cdots \times f_r) = \lambda g \circ \varphi_1 \circ (f_1 \times \cdots \times f_r) + g \circ \varphi_2 \circ (f_1 \times \cdots \times f_r).$$

Therefore, the map from the proposition is linear.

2. Let K be a field. Consider the space $K[x]_n$ of polynomials over K of degree at most n.

   (a) Find a Jordan normal form for the endomorphism

   $$\begin{array}{rccc} D: & K[x]_n & \to & K[x]_n \\ & p(x) & \mapsto & p'(x) \end{array}$$

   (b) Find a Jordan normal form for the endomorphism

   $$\begin{array}{rccc} D_2: & K[x]_n & \to & K[x]_n \\ & p(x) & \mapsto & p''(x) \end{array}$$

Solution:

(a) The matrix representation of D with respect to the standard basis $\mathcal{B}$ is

$$[D]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & n \\ & & & & 0 \end{pmatrix}$$

We read-off that its only eigenvalue is 0 and that $\dim\ker\left([D]^{\mathcal{B}}_{\mathcal{B}}\right) = 1$. Hence a Jordan normal form of D is given by the Jordan block $J_{0,n}$.

(b) We have

$$[D_2]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 2! & & & \\ & 0 & 0 & 3! & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & \frac{n!}{(n-2)!} \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

Again, we read-off that its only eigenvalue is 0 and that $\dim\ker\left([D_2]^{\mathcal{B}}_{\mathcal{B}}\right) = 2$. Hence, a Jordan normal form will comprise two blocks with 0's on the diagonal.

To figure out the size of each block, we compute the minimal polynomial of $[D_2]^{\mathcal{B}}_{\mathcal{B}}$. We know that it is a divisor of the characteristic polynomial X^n. Moreover, for $k \geqslant 1$, the matrix $([D_2]^{\mathcal{B}}_{\mathcal{B}})^k$ is the matrix representation of the endomorphism

$$\underbrace{D_2 \circ D_2 \circ \cdots \circ D_2}_{k \text{ times}} = \underbrace{D \circ D \circ \cdots \circ D}_{2k \text{ times}}.$$

Hence the smallest power for which it vanishes is $\lceil \frac{n}{2} \rceil$, which implies that the minimal polynomial of $[D_2]^{\mathcal{B}}_{\mathcal{B}}$ is $X^{\lceil \frac{n}{2} \rceil}$. This shows that the size of the biggest block in a JNF of D_2 is $\lceil \frac{n}{2} \rceil$. Since there are 2 blocks, the other block has size $\lfloor \frac{n}{2} \rfloor$.

3. Determine a Jordan normal form over $\mathbb{C}$ of the matrix

$$A := \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}$$

Lösung: We start by computing the characteristic polynomial developing with respect to the first column

$$\begin{aligned} \det(A - XI_4) &= \begin{vmatrix} -1-X & -1 & 0 & 1 \\ 0 & -X & 1 & 0 \\ 0 & -1 & -X & 0 \\ -2 & 0 & 1 & 1-X \end{vmatrix} \\ &= (-1-X) \cdot \begin{vmatrix} -X & 1 & 0 \\ -1 & -X & 0 \\ 0 & 1 & -1-X \end{vmatrix} + 2 \cdot \begin{vmatrix} -1 & 0 & 1 \\ -X & 1 & 0 \\ -1 & -X & 0 \end{vmatrix} \\ &= (X^2+1)^2 \\ &= (X-i)^2(X+i)^2. \end{aligned}$$

Hence the eigenvalues of A are $\pm i$ and both have algebraic multiplicity 2. We now compute their geometric multiplicity. We have

$$(A - iI_4) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$$\Leftrightarrow \quad \begin{cases} (-1-i)a - b + d &= 0 \\ -ib + c &= 0 \\ -b - ic &= 0 \\ -2a + c + (1-i)d &= 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} b &= 0 \\ c &= 0 \\ d &= (1+i)a \end{cases}$$

Thereupon,

$$\ker(A - iI_4) = \mathrm{Sp}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1+i \end{pmatrix}\right)$$

and $m_g(i) = 1$. Similarly, we find that $m_g(-i) = 1$.

It follows that the Jordan normal form of A consists of 2 blocks of size 2, one for each eigenvalue. It can be written explicitly as

$$\begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

4. Often the Jordan normal form is motivated by the desire to have a matrix with as much zeros as possible. Is the number of zeros actually maximized by the Jordan normal form? Stated differently: Does there exist a square matrix A over a field which has more zeros than its Jordan normal form J?

   Lösung: If A is nilpotent, the assetion is true. In this case, let J be its Jordan normal form with k Jordan blocks. Then the Eigenspace $\mathrm{Eig}_0(A) = \mathrm{Kern}(L_A)$ has dimension k, and hence A has rank $n-k$. Thus A has exactly $n-k$ linearly independent columns, hence at least $n-k$ non-zero entries. This is exactly the numbers of non-zero entries of J, As every Jordan block m to EV 0 has $m-1$ non-zero entries. Therefore, the claim is true for nilpotent matrices.

   In general, the assertion is false. A counterexample over $\mathbb{Q}$ is:

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Here A is a block upper triangular matrix consisting of 2×2-blocks, and the ones on the diagonal have characteristic polynomial $X^2 - 1 = (X-1)(X+1)$. Thus A has eigenvalues ± 1 with arithmetic multiplicity 2. Direct computations show, that every eigenvalue has geometric multiplicity 1. So A has the given JNF. It contains 10 zeros, compared to 11 zeros in A.

Remark: Even if the Jordan normal form may have less zeros as the matrix we started with, computations are still easier most of the time as the generalized eigenspaces are clear.

5. Let B be a complex 5×5-matrix with minimaly polynomial $(X-3)(X+5)^2$ and characteristic polynomial $(X-3)^2(X+5)^3$. Determine all possible Jordan normal fomrs of B.

   Lösung:

   As B has the characteristic polynomial $(X-3)^2(X+5)^3$ it has eigenvalue 3 with algebraic multiplicity 2 and eigenvalue -5 with algebraic multiplicity 3.

   The factor $(X-3)$ has power 1 in the minimal polynomial; the largest Jordan block to Eigenvalue 3 is thus a 1×1-block. Thus, the Jordan normal form contains exactly 2 Jordan blocks of size 1×1 corresponding to the eigenvalue 3.

   The factor $(X+5)$ appears in the minimal polynomail with power 2; thus there exists a Jordan block to eigenvalue -5 of size 2×2. By dimensional constraints, this yields that there exists another Jordan block of size 1×1.

   For a Jordan normal form of B we obtain up to exchanging of the Jordan blocks the only possibility
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}.$$

6. Let A be a real square matrix. We define the exponential of such a matrix as
$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$
   when it converges.

   (a) For $\lambda \in \mathbb{R}$ and $n \geqslant 1$, compute $\exp(J_{\lambda,n})$.

   (b) Determine the solution of the system of differential equations
$$\begin{aligned} x'(t) &= -x(t) + 9y(t) + 9z(t) \\ y'(t) &= 3x(t) - 6y(t) - 8z(t) \\ z'(t) &= -4x(t) + 11y(t) + 13z(t) \end{aligned}$$

with the initial conditions $x(0) = y(0) = z(0) = 1$.

Hint: Use the Jordan normal form. If you need more hints, have a look at Chapter 9.5 from Menny Akka's notes.

(c) Determine the general real solution of the differential equation

$$f^{(3)}(t) - f^{(2)}(t) + f'(t) - f(t) = 0.$$

Hint: Write the equation as a system of linear differential equations of the first order and use the Jordan normal form.

Lösung:

(a) You have seen in the lecture that for $k \geqslant 1$

$$J_{\lambda,n}^k = \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots \\ & \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots \\ & & & \binom{k}{1}\lambda^{k-1} \\ & & & \lambda^k \end{pmatrix}$$

In other words, the main diagonal consists of λ^k and the i-th diagonal over the main diagonal consists of $\binom{k}{i}\lambda^{k-i}$ as long as $i \leqslant k$. The diagonals after that vanish.

It follows that the diagonal entries of $\exp(J_{\lambda,n})$ are equal to

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}.$$

The entries on the i-th diagonal above the main diagonal are equal to

$$\begin{aligned} \sum_{k=i}^{\infty} \binom{k}{i} \frac{\lambda^{k-i}}{k!} &= \sum_{k=i}^{\infty} \frac{k!}{(k-i)!i!} \frac{\lambda^{k-i}}{k!} \\ &= \sum_{\ell=0}^{\infty} \frac{\lambda^{\ell}}{\ell!\, i!} \qquad \text{(where we set } \ell := k-i) \\ &= \frac{1}{i!} e^{\lambda}. \end{aligned}$$

(b) We set

$$v(t) := \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \qquad A := \begin{pmatrix} -1 & 9 & 9 \\ 3 & -6 & -8 \\ -4 & 11 & 13 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then the system of equations is equivalent to

$$\frac{d}{dt}v(t) = A \cdot v(t) \quad \text{with boundary condition} \quad v(0) = v\,.$$

The unique solution is $v(t) = \exp(At) \cdot v_0$.
To determine it explicitly, we transform A into Jordan normal form. The characteristic polynomial of A is

$$\mathrm{char}_A(X) \;=\; X^3 - 6X^2 + 12X - 8 \;=\; (X-2)^3\,.$$

For $B := A - 2I_3$ we have

$$B = \begin{pmatrix} -3 & 9 & 9 \\ 3 & -8 & -8 \\ -4 & 11 & 11 \end{pmatrix}, \qquad B^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 3 & 3 \\ 1 & -3 & -3 \end{pmatrix}$$

and $B^k = 0$ for all $k \geqslant 3$. We choose any $w \in \mathbb{R}^3 \smallsetminus \mathrm{Kern}(B^2)$, e.g.

$$w := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then the vectors w, Bw, B^2w form a basis of $\mathbb{R}^3$ and with

$$S := (B^2w, Bw, w) = \begin{pmatrix} 0 & -3 & 1 \\ -1 & 3 & 0 \\ 1 & -4 & 0 \end{pmatrix}$$

we get the following decomposition of A:

$$A = S \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \cdot S^{-1}\,.$$

The solution of exercise 1 of sheet 15 yields for all $k \geqslant 0$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 2^k & \binom{k}{1}2^{k-1} & \binom{k}{2}2^{k-2} \\ 0 & 2^k & \binom{k}{1}2^{k-1} \\ 0 & 0 & 2^k \end{pmatrix}.$$

Plugging it into the exponential series yields

$$\exp\Bigg(\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} t\Bigg) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}^k t^k \;=\; \begin{pmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix},$$

and hence

$$
\begin{aligned}
v(t) \;=\; \exp(At)\cdot v_0 \;&=\; S\cdot\exp\Bigg(\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}t\Bigg)\cdot S^{-1}v\\
&= S\cdot\exp\Bigg(\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}t\Bigg)\cdot\begin{pmatrix}-7\\-2\\-5\end{pmatrix}\\
&= -S\cdot\left(e^{2t}\begin{pmatrix}7\\2\\5\end{pmatrix}+te^{2t}\begin{pmatrix}2\\5\\0\end{pmatrix}+t^2e^{2t}\begin{pmatrix}[r]5/2\\0\\0\end{pmatrix}\right)\\
&= e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix}+te^{2t}\begin{pmatrix}15\\-13\\18\end{pmatrix}+t^2e^{2t}\begin{pmatrix}0\\5/2\\-5/2\end{pmatrix}
\end{aligned}
$$

(c) We set

$$
F(t) := \begin{pmatrix}f(t)\\f'(t)\\f''(t)\end{pmatrix}\qquad\text{and}\quad A := \begin{pmatrix}0&1&0\\0&0&1\\1&-1&1\end{pmatrix};
$$

then the differential equation from the exercise is equivalent to

$$
\frac{d}{dt}F(t) = A\cdot F(t)\,.
$$

The solution of this equation with arbitrary boundary condition

$$
F(0) = \begin{pmatrix}f(0)\\f'(0)\\f''(0)\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = v_0
$$

then is $F(t) = \exp(At)\cdot v_0$, and the general solution for $f(t)$ is the first entry of $F(t)$.

Now we compute the characteristic polynomial of A and find

$$
\mathrm{char}_A(X) \;=\; (X-1)(X^2+1)\,.
$$

Hence A is a 3×3-matrix with 3 different complex eigenvalues 1 and $\pm i$ ans so is diagonalizable over $\mathbb{C}$. We continue by doing computations over $\mathbb{C}$ and will only in the end reduce to $\mathbb{R}$. For this, we write $A = UJU^{-1}$ with a matrix $U \in \mathrm{GL}_3(\mathbb{C})$ and

$$
J := \begin{pmatrix}1&0&0\\0&i&0\\0&0&-i\end{pmatrix}.
$$

Then we have

$$\exp(At)\cdot v_0 \;=\; \exp(U\cdot Jt\cdot U^{-1})\cdot v_0 \;=\; U\cdot \exp(Jt)\cdot U^{-1}v_0.$$

The exponential series behaves well with the diagonal matrix Jt and we get

$$\exp(Jt) \;=\; \exp\Bigg(\begin{pmatrix} t & 0 & 0\\ 0 & it & 0\\ 0 & 0 & -it\end{pmatrix}\Bigg) \;=\; \Bigg(\begin{pmatrix} \exp(t) & 0 & 0\\ 0 & \exp(it) & 0\\ 0 & 0 & \exp(-it)\end{pmatrix}\Bigg).$$

Hence the first component of $\exp(At)\cdot v_0$ is a linear combination of $\exp(t)$ and $\exp(\pm it)$ with constant coefficients in $\mathbb{C}$. As $\exp(\pm it) = \cos(t)\pm i\sin(t)$, it is equivalently a linear combination of $\exp(t)$, $\cos(t)$, and $\sin(t)$ with constant coefficient in $\mathbb{C}$. For the function to have real values, these coefficients must lie in $\mathbb{R}$. Hence every real solution is of the form

$$f(t) \;=\; ae^t + b\cos(t) + c\sin(t)$$

for constants $a, b, c \in \mathbb{R}$. On the other hand, direct computations show that every such function is a solution.

Aliter for (b): Computation of a Jordan basis of $\mathbb{R}^3$ with respect to A yields a Jordan normal form of A over $\mathbb{R}$:

$$A = \begin{pmatrix} 1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & 0 & 1\\ 1 & 0 & -1\\ -1 & 2 & -1\end{pmatrix}$$

By induction we find that for all $m \geqslant 0$:

$$\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}^{2m} = (-1)^k I_2 \quad\text{and}\quad \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}^{2m+1} = (-1)^m\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

Plugging this into the matrix exponential yields

$$\begin{aligned}
\exp\left(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}t\right) &= \sum_{m=0}^{\infty}\frac{t^{2m}}{(2m)!}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}^{2m} + \sum_{m=0}^{\infty}\frac{t^{2m+1}}{(2m+1)!}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}^{2m+1}\\
&= \Big(\sum_{m=0}^{\infty}\frac{(-1)^m t^{2m}}{(2m)!}\Big)\cdot I_2 + \Big(\sum_{m=0}^{\infty}\frac{(-1)^m t^{2m+1}}{(2m+1)!}\Big)\cdot\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\\
&= \cos(t)\cdot I_2 + \sin(t)\cdot\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.
\end{aligned}$$

Together, this show

$$\begin{aligned}
\exp(At) &= \begin{pmatrix} 1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix} e^t & 0 & 0\\ 0 & \cos(t) & \sin(t)\\ 0 & -\sin(t) & \cos(t)\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & 0 & 1\\ 1 & 0 & -1\\ -1 & 2 & -1\end{pmatrix}\\
&= \frac{e^t}{2}\begin{pmatrix} 1 & 0 & 1\\ 1 & 0 & 1\\ 1 & 0 & 1\end{pmatrix} + \frac{\cos(t)}{2}\begin{pmatrix} 1 & 0 & -1\\ -1 & 2 & -1\\ -1 & 0 & 1\end{pmatrix} + \frac{\sin(t)}{2}\begin{pmatrix} -1 & 2 & -1\\ -1 & 0 & 1\\ 1 & -2 & 1\end{pmatrix}.
\end{aligned}$$

The general solution $f(t)$ is the first component of $\exp(At) \cdot v_0$, i.e.

$$f(t) \;=\; \frac{1}{2}(x_1 + x_3)e^t + \frac{1}{2}(x_1 - x_3)\cos(t) + \frac{1}{2}(-x_1 + 2x_2 - x_3)\sin(t).$$