Musterlösung Serie 26

Multilinear algebra, Tensor product

1. Simplify the following expression in $\mathbb{R}^2 \otimes \mathbb{R}^3$.

$$-\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}3\\-1\\2\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix} \otimes \begin{pmatrix}1\\0\\1\end{pmatrix} + 2\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}2\\-1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}5\\0\\1\end{pmatrix}$$

Lösung: We express the first factor in each summand as a linear combination of the standard basis vectors $\binom{1}{0}$ und $\binom{0}{1}$, use that the tensor product is linear in both variables. This yields

$$\begin{aligned} -\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}3\\-1\\2\end{pmatrix} &= \left((-1)\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix}\right) \otimes \begin{pmatrix}3\\-1\\2\end{pmatrix} \\ &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}-3\\1\\-2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}-6\\2\\-4\end{pmatrix}. \end{aligned}$$

In the same way, we get.

$$\begin{aligned} \begin{pmatrix}3\\2\end{pmatrix} \otimes \begin{pmatrix}1\\0\\1\end{pmatrix} &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}3\\0\\3\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}2\\0\\2\end{pmatrix} \\ 2\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}2\\-1\\1\end{pmatrix} &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}4\\-2\\2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}4\\-2\\2\end{pmatrix}. \end{aligned}$$

The desired expression thus simplifies to

$$\begin{aligned}
& \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}-3\\1\\-2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}-6\\2\\-4\end{pmatrix} \\
+ & \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}3\\0\\3\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}2\\0\\2\end{pmatrix} \\
+ & \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}4\\-2\\2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}4\\-2\\2\end{pmatrix} \\
- & \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}5\\0\\1\end{pmatrix} \\
= & \begin{pmatrix}1\\0\end{pmatrix} \otimes \left(\begin{pmatrix}-3\\1\\-2\end{pmatrix} + \begin{pmatrix}3\\0\\3\end{pmatrix} + \begin{pmatrix}4\\-2\\2\end{pmatrix} - \begin{pmatrix}5\\0\\1\end{pmatrix} \right) \\
+ & \begin{pmatrix}0\\1\end{pmatrix} \otimes \left(\begin{pmatrix}-6\\2\\-4\end{pmatrix} + \begin{pmatrix}2\\0\\2\end{pmatrix} + \begin{pmatrix}4\\-2\\2\end{pmatrix} \right) \\
= & \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}-1\\-1\\2\end{pmatrix}
\end{aligned}$$

2. Let K be a field. Consider the bilinear map

$$\begin{array}{rccc} \psi : & K^m_{\text{cols}} \times K^n_{\text{cols}} & \to & M_{m\times n}(K) \\ & (u, v) & \mapsto & u \cdot v^T \end{array}$$

   (a) Prove that the image of ψ is the set of matrices of rank lower or equal to 1.
   (b) Is $\text{Im}(\psi)$ a linear subspace?
   (c) Describe $\text{Sp}(\text{Im}(\psi))$.

Solution: Let us denote $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_n)$. Then

$$u \cdot v^T = \begin{pmatrix} v_1u_1 & v_2u_1 & \cdots & v_nu_1 \\ v_1u_2 & v_2u_2 & \cdots & v_nu_2 \\ \vdots & \vdots & & \vdots \\ v_1u_m & v_2u_m & \cdots & v_nu_m \end{pmatrix}$$

(a) We directly see that each column of $\psi(u, v)$ is a multiple of u. Hence

$$\operatorname{rank}(\psi(u, v)) \leqslant 1.$$

Conversely, let $A \in M_{m\times n}(K)$ be a matrix of rank lower or equal to 1. Then each of the column of A is a multiple of a vector $u_0 \in K^m_{\text{cols}}$, say the i-th column of A equals $a_i u_0$ for $a_i \in K$. Then

$$A = \psi(u_0, (a_i)_{i=1}^n) \in \operatorname{Im}(\psi).$$

(b) It isn't. Indeed consider the two matrices of rank one

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The sum $B_1 + B_2 = I_2$ is of rank 2 and therefore is not in $\operatorname{Im}(\psi)$.

(c) For $1 \leqslant k \leqslant m$ and $1 \leqslant \ell \leqslant n$, define $E_{k\ell} \in M_{m\times n}(K)$ as follows:

$$(E_{k\ell})_{ij} = \begin{cases} 1, & \text{if } i = k \wedge j = \ell \\ 0, & \text{otherwise} \end{cases}$$

The set

$$\mathcal{B} := \{E_{k\ell} \mid 1 \leqslant k \leqslant m,\ 1 \leqslant \ell \leqslant n\}$$

is both a subset of $\operatorname{Im}(\psi)$ and a basis of $M_{m\times n}(K)$. Hence

$$\operatorname{Sp}(\operatorname{Im}(\psi)) = M_{m\times n}(K).$$

3. Let V, W be finite-dimensional vector spaces over a field K. Show that

$$V^* \otimes W \cong \operatorname{Hom}(V, W).$$

Solution: Define the map

$$\begin{array}{rccc} \varphi: & V^* \times W & \to & \operatorname{Hom}(V, W) \\ & (\ell, w) & \mapsto & (\varphi_{(\ell,w)} : v \mapsto \ell(v)w) \end{array}$$

It follows from the rules for addition and scalar multiplication for linear functionals and linear maps that this map is bilinear (check it!). Hence there exists a unique linear map $\eta : V^* \otimes W \to \operatorname{Hom}(V, W)$ such that the diagram

$$\begin{array}{ccc} V^* \times W & \xrightarrow{\ \tau\ } & V^* \otimes W \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \eta} \\ & & \operatorname{Hom}(V, W) \end{array}$$

commutes. In other words, for any $\ell \in V^*$, $w \in W$, we have

$$\eta(\ell \otimes w) = \varphi_{(\ell,w)}.$$

We first show that this map is surjective. Let $L \in \mathrm{Hom}(V, W)$. Let $\mathcal{B} = \{e_i\}_{i=1}^n$ be a basis of V and let $\mathcal{B}^*$ be the corresponding dual basis. Consider $\{w_i := L(e_i) \mid 1 \leqslant i \leqslant n\} \subset W$. Now, for any $v \in V$,

$$\eta\left(\sum_{i=1}^n e_i^* \otimes w_i\right)(v) = \sum_{i=1}^n \varphi_{(e_i^*, w_i)}\left(\sum_{j=1}^n e_j^*(v) e_j\right) = \sum_{i=1}^n e_i^*(v) L(e_i) = L(v).$$

It follows that η is surjective.

We now show that η is injective. Denote $\mathcal{C} = \{f_j\}_{j=1}^m$ a basis of W. Assume that

$$\eta\left(\sum_{i=1}^n \sum_{j=1}^m \alpha_{ij}(e_i^* \otimes f_j)\right) = \eta\left(\sum_{i=1}^n \sum_{j=1}^m \beta_{ij}(e_i^* \otimes f_j)\right).$$

This is equivalent to

$$\begin{aligned}
\forall v \in V: &\quad \sum_{i=1}^n \sum_{j=1}^m \alpha_{ij} e_i^*(v) f_j = \sum_{i=1}^n \sum_{j=1}^m \beta_{ij} e_i^*(v) f_j \\
\Leftrightarrow \forall v \in V,\ \forall 1 \leqslant j \leqslant m: &\quad \sum_{i=1}^n \alpha_{ij} e_i^*(v) = \sum_{i=1}^n \beta_{ij} e_i^*(v) \\
\Leftrightarrow \forall 1 \leqslant j \leqslant m: &\quad \sum_{i=1}^n \alpha_{ij} e_i^* = \sum_{i=1}^n \beta_{ij} e_i^* \\
\Leftrightarrow \forall 1 \leqslant j \leqslant m,\ \forall 1 \leqslant i \leqslant n: &\quad \alpha_{ij} = \beta_{ij},
\end{aligned}$$

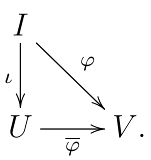
which yields

$$\sum_{i=1}^n \sum_{j=1}^m \alpha_{ij}(e_i^* \otimes f_j) = \sum_{i=1}^n \sum_{j=1}^m \beta_{ij}(e_i^* \otimes f_j).$$

It follows that η is injective, which concludes the proof.

4. Consider a set I. For a pair (U, ι) consisting of a K-vectorspace U and a map $\iota\colon I \to U$ consider the following *universal property*:

   For every K-vectorspace V and for every map $\varphi\colon I \to V$ there exists exactly one linear map $\overline{\varphi}\colon U \to V$, such that the following diagram commutes:



(a) Show that for two pairs (U, ι) and (U', ι') satisfying the universal property, there exists a unique isomorphism $\psi \colon U \xrightarrow{\sim} U'$ with $\psi \circ \iota = \iota'$.

(b) Show that the universal property is satisfied for the K-vectorspace

$$K^{(I)} := \big\{(x_i)_{i\in I} \in K^I \;\big|\; x_i \neq 0 \text{ for at most finitely many } i\big\}$$

with the map

$$\iota_I : I \to K^{(I)}, \quad i \mapsto (\delta_{ij})_{j\in I}\,.$$

Lösung:

(a) Die universelle Eigenschaft von (U, ι) angewandt auf die Abbildung $\iota' \colon I \to U'$ ergibt eine eindeutige lineare Abbildung $\psi \colon U \to U'$ mit $\psi\circ\iota = \iota'$. Wir zeigen, dass ψ ein Isomorphismus ist. Die universelle Eigenschaft von (U', ι') angewandt auf die Abbildung $\iota \colon I \to U$ ergibt eine eindeutige lineare Abbildung $\varphi \colon U' \to U$ so dass $\varphi \circ \iota' = \iota$ ist. Es folgt

$$\begin{aligned} \varphi\circ\psi\circ\iota &= \varphi\circ\iota' = \iota = \mathrm{id}_U \circ\iota \qquad \text{und} \\ \psi\circ\varphi\circ\iota' &= \psi\circ\iota = \iota' = \mathrm{id}_{U'} \circ\iota. \end{aligned}$$

Die Eindeutigkeit in der universellen Eigenschaft von (U, ι) angewandt auf $\varphi \circ \psi$ impliziert nun $\varphi \circ \psi = \mathrm{id}_U$, und die Eindeutigkeit in der universellen Eigenschaft von (U', ι') angewandt auf $\psi\circ\varphi$ zeigt $\psi\circ\varphi = \mathrm{id}_{U'}$. Also ist φ ein beidseitiges Inverses von ψ, und somit ist ψ ein Isomorphismus.

(b) Betrachte eine beliebige Abbildung $\varphi \colon I \to V$. Aus dem Herbstsemester wissen wir, dass die Teilmenge $B := \{\, \iota_I(i) \,|\, i \in I \,\}$ eine Basis von $K^{(I)}$ ist. Die gesuchte lineare Abbildung $\overline{\varphi} \colon K^{(I)} \to V$ ist also eindeutig bestimmt durch ihre Einschränkung auf B, oder äquivalent durch die Komposition $\overline{\varphi}\circ i_I = \varphi$. Umgekehrt ist

$$\overline{\varphi} \colon K^{(I)} \to V,\ (x_i)_{i\in I} \mapsto \sum_{i\in I} x_i\varphi(i)$$

eine lineare Abbildung mit $\overline{\varphi}(\iota_I(i)) = \varphi(i)$ für alle $i \in I$. Damit sind die Eindeutigkeit und die Existenz gezeigt.

5. Prove the following statements concerning vector spaces U, V, V_1, V_2 over a field K:

(a) There is a unique linear map $\kappa : U \otimes V \to V \otimes U$ that satisfies

$$\kappa(u \otimes v) = v \otimes u.$$

This map is an isomorphism.

(b) There is a canonical isomorphism between $U \otimes K$ and U.

(c) There is a canonical isomorphism

$$U \otimes (V_1 \oplus V_2) \to (U \otimes V_1) \oplus (U \otimes V_2).$$

Remark. Do not define the required homomorphisms with respect to bases.

Solution:

(a) Let $\varphi : U \times V \to V \otimes U$, $(u, v) \mapsto v \otimes u$ and $\tilde{\varphi} : V \times U \to U \otimes V$, $(v, u) \mapsto u \otimes v$. These are bilinear maps; hence there exist unique linear maps κ and $\tilde{\kappa}$ such that the following diagrams commute

$$\begin{array}{ccc} U \times V & \xrightarrow{\tau} & U \otimes V \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \exists!\ \kappa} \\ & & V \otimes U \end{array} \qquad\qquad \begin{array}{ccc} V \times U & \xrightarrow{\tilde{\tau}} & V \otimes U \\ & \searrow^{\tilde{\varphi}} & \downarrow{\scriptstyle \exists!\ \tilde{\kappa}} \\ & & U \otimes V \end{array}$$

We compute that

$$\kappa(u \otimes v) = \kappa(\tau(u, v)) = \varphi(u, v) = v \otimes u$$
$$\tilde{\kappa}(v \otimes u) = \tilde{\kappa}(\tilde{\tau}(v, u)) = \tilde{\varphi}(v, u) = u \otimes v.$$

Hence κ and $\tilde{\kappa}$ are mutual inverses, which shows that κ is the isomorphism we were looking for.

(b) Let W be a K-vector space and let $\psi : U \times K$ be a bilinear map. Define $\varphi : U \times K \to U$, $(u, k) \mapsto ku$, and $\eta : U \to W$ such that $\eta(u) = \psi(u, 1)$. We compute that

$$\eta \circ \varphi(u, k) = \eta(ku) = \psi(ku, 1) = k\psi(u, 1) = \psi(u, k),$$

where we used that ψ is bilinear. Thereupon the following diagram commutes

$$\begin{array}{ccc} U \times K & \xrightarrow{\varphi} & U \\ & \searrow^{\psi} & \downarrow{\scriptstyle \eta} \\ & & W \end{array}$$

Moreover, η is a linear map since for any $u_1, u_2 \in U$ and for any $\alpha \in K$,

$$\eta(u_1 + \alpha u_2) = \psi(u_1 + \alpha u_2, 1) = \psi(u_1, 1) + \alpha\psi(u_2, 1) = \eta(u_1) + \alpha\eta(u_2)$$

by bilinearity of ψ. It is also unique since for any linear map $\tilde{\eta}$ such that

$$\begin{array}{ccc} U \times K & \xrightarrow{\varphi} & U \\ & \searrow^{\psi} & \downarrow{\scriptstyle \tilde{\eta}} \\ & & W \end{array}$$

commutes, we have, for any $k \in K^*$, for any $u \in U$,

$$\begin{aligned} & k\tilde{\eta}(u) = \tilde{\eta}(ku) = e\tilde{t}a(\varphi(u, k)) = \psi(u, k) = k\eta(u) \\ \implies \quad & \tilde{\eta}(u) = \eta(u). \end{aligned}$$

So, the pair (U, φ) is a tensor product for $U \times K$ and is therefore isomorphic to $U \otimes K$.

(c) We define a map

$$\begin{array}{rccc} \varphi : & U \times (V_1 \oplus V_2) & \to & (U \otimes V_1) \oplus (U \otimes V_2) \\ & (u, (v_1, v_2)) & \mapsto & (u \otimes v_1, u \otimes v_2) \end{array}$$

This map is linear in both U and $V_1 \oplus V_2$ by properties of the tensor product (check this!). Hence it induces a unique linear map $\eta : U \otimes (V_1 \oplus V_2) \to (U \otimes V_1) \oplus (U \otimes V_2)$ such that the diagram

$$\begin{array}{ccc} U \times (V_1 \oplus V_2) & \xrightarrow{\ \tau\ } & U \otimes (V_1 \oplus V_2) \\ & \searrow^{\varphi} & \downarrow \eta \\ & & (U \otimes V_1) \oplus (U \otimes V_2) \end{array}$$

commutes. In other words, for any $u \in U$, $v_1 \in V_1$, $v_2 \in V_2$, we have

$$\eta(u \otimes (v_1, v_2)) = (u \otimes v_1, u \otimes v_2).$$

This map is clearly surjective.
Assume that elements

$$A = \sum_{i=1}^{k} \alpha_i (u_i \otimes (v_1^{(i)}, v_2^{(i)})),\ \ B = \sum_{j=1}^{\ell} \beta_j (\tilde{u}_j \otimes (\tilde{v}_1^{(j)}, \tilde{v}_2^{(j)})) \in U \otimes (V_1 \oplus V_2),$$

we have $\eta(A) = \eta(B)$. This is equivalent to

$$\sum_{i=1}^{k} \alpha_i (u_i \otimes v_1^{(i)}, u_i \otimes v_2^{(j)}) = \sum_{j=1}^{\ell} \beta_j (\tilde{u}_j \otimes \tilde{v}_1^{(j)}, \tilde{u}_j \otimes \tilde{v}_2^{(j)})$$

$$\iff \quad \left\{ \begin{array}{rcl} \sum_{i=1}^{k} \alpha_i (u_i \otimes v_1^{(i)}) & = & \sum_{j=1}^{\ell} \beta_j (\tilde{u}_j \otimes \tilde{v}_1^{(j)}) \\ \sum_{i=1}^{k} \alpha_i (u_i \otimes v_2^{(i)}) & = & \sum_{j=1}^{\ell} \beta_j (\tilde{u}_j \otimes \tilde{v}_2^{(j)}) \end{array} \right.$$

Since the expressions on both sides of the equations above are written as a linear combination of pure tensors, which constitute a basis of $U \otimes V_1$, respectively $U \otimes V_2$, we must have $k = \ell$ and for each $i \in \{1, \ldots, k\}$ there exists $j \in \{1, \ldots, \ell\}$ such that

$$\left\{ \begin{array}{rcl} u_i & = & \tilde{u}_j \\ \alpha_i & = & \beta_j \\ v_1^{(i)} & = & \tilde{v}_1^{(j)} \\ v_2^{(i)} & = & \tilde{v}_2^{(j)} \end{array} \right.$$

Thereupon $A = B$, and it follows that η is injective, which concludes the proof.

6. Let V be a vector space of dimension $n < \infty$, and let f be an endomorphism of V with characteristic polynomial $\mathrm{char}_f(X) = \sum_{i=0}^n a_i X^i$. For all $r > 0$ consider the induced map

$$\mathrm{Alt}^r(f)\colon\ \mathrm{Alt}^r_K(V,K) \to \mathrm{Alt}^r_K(V,K),\ \varphi \mapsto \varphi \circ (f \times \ldots \times f).$$

Show: For all $r = 1, \ldots, n$, we have

$$a_{n-r} = (-1)^{n+r}\,\mathrm{Tr}\,\mathrm{Alt}^r(f).$$

Lösung: Sei $B := (b_1, \ldots, b_n)$ eine geordnete Basis von V und sei

$$A := (a_{ij})_{1 \leqslant i,j \leqslant n} := {}_B M_B(f)$$

für Koeffizienten $a_{ij} \in K$ die Darstellungsmatrix von f bezüglich B. Für alle Indizes $1 \leqslant i_1 < \ldots < i_r \leqslant n$ sei

$$\varphi_{(i_1,\ldots,i_r)} : V \times \ldots \times V \to K$$

die eindeutige alternierende Abbildung mit

$$\varphi_{(i_1,\ldots,i_r)}(b_{j_1}, \ldots, b_{j_r}) = \begin{cases} \mathrm{sgn}(\sigma) & \text{falls } (j_1,\ldots,j_r) = (i_{\sigma(1)},\ldots,i_{\sigma(r)}) \text{ für ein } \sigma \in S_r \\ 0 & \text{sonst} \end{cases}$$

für alle $1 \leqslant j_1, \ldots, j_r \leqslant n$. Aus Kapitel 12.1 und 12.2 der Zusammenfassung und durch direktes Verifizieren folgt, dass

$$\left\{\varphi_{(i_1,\ldots,i_r)} \mid 1 \leqslant i_1 < \ldots < i_r \leqslant n\right\}$$

eine Basis von $\mathrm{Alt}^r_K(V,K)$ bildet. Ein beliebiges Element $\varphi \in \mathrm{Alt}^r_K(V,K)$ ist dann die Linearkombination

$$\sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \varphi(b_{i_1}, \ldots, b_{i_r}) \cdot \varphi_{(i_1,\ldots,i_r)}.$$

Mit dieser Formel und der Definition von $\varphi_{(i_1,\ldots,i_r)}$ berechnen wir:

$$\begin{aligned}
\mathrm{Tr}\,\mathrm{Alt}^r(f) &= \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \mathrm{Alt}^r(f)(\varphi_{(i_1,\ldots,i_r)})(b_{i_1}, \ldots, b_{i_r}) \\
&= \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \varphi_{(i_1,\ldots,i_r)}(f(b_{i_1}), \ldots, f(b_{i_r})) \\
&= \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \ \sum_{k_1,\ldots,k_r=1}^{n} a_{k_1 i_1} \ldots a_{k_r i_r} \varphi_{(i_1,\ldots,i_r)}(b_{k_1}, \ldots, b_{k_r}) \\
&= \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \ \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) a_{i_{\sigma(1)} i_1} \ldots a_{i_{\sigma(r)} i_r} \\
&= \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \ \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) a_{i_1 i_{\sigma(1)}} \ldots a_{i_r i_{\sigma(r)}} t
\end{aligned}$$

Andererseits gilt für das charakteristische Polynom von f

$$\begin{aligned}
\mathrm{char}_f(X) &= \det(X \cdot I_n - A) \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot (X \cdot \delta_{1\sigma(1)} - a_{1\sigma(1)}) \cdots (X \cdot \delta_{n\sigma(n)} - a_{n\sigma(n)}) \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \sum_{r=0}^{n} \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (-a_{i_1\sigma(i_1)}) \cdots (-a_{i_r\sigma(i_r)}) \cdot \prod_{\substack{1 \leqslant j \leqslant n \\ j \notin \{i_1, \ldots, i_r\}}} (X \cdot \delta_{j\sigma(j)}) \\
&= \sum_{r=0}^{n} (-1)^r X^{n-r} \cdot \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{i_1\sigma(i_1)} \cdots a_{i_r\sigma(i_r)} \cdot \prod_{j \notin \{i_1, \ldots, i_r\}} \delta_{j\sigma(j)} \\
&= \sum_{r=0}^{n} (-1)^r X^{n-r} \cdot \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} \sum_{\tau \in S_r} \mathrm{sgn}(\tau) a_{i_1 i_{\tau(1)}} \cdots a_{i_r i_{\tau(r)}}.
\end{aligned}$$

Durch Vergleichen des Koeffizienten von X^{n-r} mit unserem Ausdruck für $\mathrm{Tr}\,\mathrm{Alt}^r(f)$ erhalten wir

$$\sum_{r=0}^{n} a_{n-r} X^{n-r} = \mathrm{char}_f(X) = \sum_{r=0}^{n} (-1)^r \,\mathrm{Tr}(\mathrm{Alt}^r(f)) X^{n-r},$$

also die Aussage der Aufgabe.