Musterlösung Serie 13

This exercise sheet is to be handed in the week before the Frühjahr Semester 2023.

1. Consider the linear subspace

$$U := \left\langle (2,2,2,2,2)^T, (1,2,2,2,2)^T, (1,1,2,2,2)^T \right\rangle$$

of $V := \mathbb{R}^5$. Determine a subset of the standard basis of $\mathbb{R}^5$, which maps bijectively to a basis of V/U.

Solution: The subset needs to be the basis of a complement of U. Trial and error finds for example the solution

$$\left\{ (0,0,1,0,0)^T, (0,0,0,1,0)^T \right\}$$

2. Let V, W be vector spaces over a field K. Let $U \subseteq V$ a linear subspace, and consider a linear map $f : V \to W$ with $U \subseteq \operatorname{Ker}(f)$. Moreover, consider the linear map induced by the universal property of the quotient vector space

$$\begin{array}{rccc} \bar{f}: & V/U & \to & W \\ & v+U & \to & f(v) \end{array}$$

Show:

(a) $\operatorname{Ker}(\bar{f}) = \operatorname{Ker}(f)/U$.

(b) $\bar{f}$ is injective iff $U = \operatorname{Ker}(f)$.

(c) $\bar{f}$ is surjective iff f is surjective.

(d) If f is surjective, then f induces an isomorphism $V/\operatorname{Ker}(f) \xrightarrow{\sim} W$.

The abbreviation 'iff' is short for 'if and only if' and is very common in mathematical texts. Solution:

(a) The map $\bar{f} : V/U \to W$ satisfies $\bar{f}(x+U) = f(x)$ for all $x \in V$. This yields

$$\begin{aligned} \operatorname{Ker}(\bar{f}) &= \{v+U \mid v \in V \wedge \bar{f}(v+U) = 0\} \\ &= \{v+U \mid v \in V \wedge f(v) = 0\} \\ &= \{v+U \mid v \in \operatorname{Ker}(f)\} \\ &= \operatorname{Ker}(f)/U \end{aligned}$$

(b) $\bar{f}$ is injektive $\Longleftrightarrow \operatorname{Ker}(\bar{f}) = 0 \Longleftrightarrow \operatorname{Ker}(f)/U = 0 \Longleftrightarrow \operatorname{Ker}(f) = U$.

(c) We have

$$\begin{aligned}\operatorname{Im}(f) &= \{w \in W \mid \exists v \in V : f(v) = w\} \\ &= \{w \in W \mid \exists v \in V : \bar{f}(v + U) = w\} \\ &= \{w \in W \mid \exists \bar{v} \in V/U : \bar{f}(\bar{v}) = w\} \\ &= \operatorname{Im}(\bar{f}).\end{aligned}$$

(d) The induced map $\bar{f} : V/\operatorname{Ker}(f) \to W$ iis injective because of (b) and surjective because of (c). Thus, it is an isomorphism.

3. Suppose T is a function from V to W. The graph of T, denoted $\Gamma(T)$, is the subset of $V \oplus W$ defined by

$$\Gamma(T) = \{(v, Tv) \in V \oplus W : v \in V\}$$

Prove that T is a linear map if and only if the graph of T is a linear subspace of $V \oplus W$.

Remark. Formally, a function T from V to W is a subset T of $V \oplus W$ such that for each $v \in V$, there exists exactly one element $(v, w) \in T$. In other words, formally a function is what is called above its graph. We do not usually think of functions in this formal manner. However, if we do become formal, then the exercise above could be rephrased as follows: Prove that a function T from V to W is a linear map if and only if T is a subspace of $V \oplus W$.

Solution: First assume that $T : V \to W$ is linear. Let $\mu \in K$, and let $v, w \in V$. Then $(v, Tv), (w, Tw) \in \Gamma(T)$. Moreover, we have

$$\begin{aligned}(v, Tv) + \mu(w, Tw) &= (v, Tv) + (\mu w, \mu Tv) \\ &= (v + \mu w, Tv + \mu Tw) \\ &= (v + \mu w, T(v + \mu w)) \in \Gamma(T),\end{aligned}$$

where we used that T is linear to obtain the last equality. This shows that $\Gamma(T)$ is a subspace of $V \oplus W$.

On the other hand, assume that $\Gamma(T)$ is a linear subspace of $V \oplus W$. We let $\mu \in K$ and $v, w \in V$. Then (v, Tv) and (w, Tw) are elements of $\Gamma(T)$, hence, by linearity,

$$\Gamma(T) \ni (v, Tv) + \mu(w, Tw) = (v + \mu w, Tv + \mu Tw).$$

By definition of $\Gamma(T)$, this implies that

$$T(v + \mu w) = Tv + \mu Tw.$$

We conclude that T is linear.

4. Let V be a finite-dimensional vector space over a field K.

(a) Let $U \subseteq V$ be a subspace and denote W one of its linear complements. Define an isomorphism between V and $U \oplus W$.

(b) Show that any linear map $\alpha : U \to K$ can be extended to a linear map $\tilde{\alpha}$ on the whole of V. Does $\tilde{\alpha}$ depend on a choice of complement of U?

(c) Define an isomorphism

$$V^* \cong U^* \oplus W^*.$$

Solution:

(a) By properties of the linear complement, we know that for any $v \in V$ there exist a unique $u_v \in U$ and a unique $w_v \in W$ such that $v = u_v + w_v$. Define a map

$$\begin{array}{rccc} \Phi : & V & \to & U \oplus V \\ & v & \mapsto & (u_v, w_v) \end{array}$$

A direct computation shows that this map is linear. Since u_v and w_v are uniquely determined by v, this map is injective. Let $(u, w) \in U \oplus W$. Then

$$(u, w) = \Phi(u + w),$$

which shows that Φ is surjective.

(b) Let $\alpha : U \to K$ be an arbitrary linear map. For each subspace U choose a complement U' in V and define the map $\tilde{\alpha} : V \to K$ by

$$\tilde{\alpha}(v) := \alpha\,(u)$$

for every $v = u + u'$ with $u \in U$ and $u' \in U'$. As U' is a linear complementt of U, we get that $\tilde{\alpha}$ is well defined. Linearity of $\tilde{\alpha}$ can be verified by explicit strait forward combinations In other words, this means $\tilde{\alpha} \in V^*$. From $\tilde{\alpha}|_U = \alpha$ the claim follows.

(c) For every linear map $\ell : V \to K$, the restrictions $\ell|_U : U \to K$ and $\ell|_W : W \to K$ are again linear. Hence we have a well defined map

$$\psi : V^* \to U^* \oplus W^*, \quad \ell \mapsto \left(\ell|_U\, , \ell|_W\right).$$

Direct computations show that ψ is linear. As $V = U + W$, we have that any linear map $\ell : V \to K$ is determined by its restrictions to U and W. Hence ψ is injective.

Now consider any linear maps $\ell_1 : U \to K$ and $\ell_2 : W \to K$. As every $v \in V$ can be written as the sum $u + w$ with $u \in U$ and $w \in W$ in a unique way, we can construct a well defined map $\ell : V \to K$ by $\ell\,(u + w) = \ell_1\,(u) + \ell_2\,(w)$. Direct computations show that ℓ is linear. The construction also yields $\psi(\ell) = (\ell_1, \ell_2)$. Hence ψ is surjektive. Together we have that ψ is a bijective linear map and therefore an isomorphism.

5. Let $(v_1, \ldots, v_n)$ be an ordered basis of a vector space V, and let $(v_1^*, \ldots, v_n^*)$ the corresponding dual basis of the dual vector spaceV^*, and let $(v_1^{**}, \ldots, v_n^{**})$ the dual basis corresponding to B^* of the bidual vecot space $(V^*)^*$. Show, that the natural isomorphism
$$\tau : V \xrightarrow{\sim} (V^*)^*, \quad v \mapsto \tau(v)$$
maps every v_j to the corresponding v_j^{**}.

Lösung: The dual basis is characterised by the condition $v_i^*(v_j) = \delta_{i,j}$ for all i, j. Similarly, we have $v_j^{**}(v_i^*) = \delta_{i,j}$ for all i, j. Moreover, the evaluation map is defined by $\tau(v)(\ell) = \ell(v)$ for all $v \in V$ and $\ell \in V^*$. For every pair i, j, this yields
$$\tau(v_j)(v_i^*) = v_i^*(v_j) = \delta_{i,j} = v_j^{**}(v_i^*).$$
Hence the linear maps $\tau(v_j) : V^* \to K$ and $v_j^{**} : V^* \to K$ are equal on the bases $(v_1^*, \ldots, v_n^*)$ of V^* and are thus equal.

6. Let U, V, W_1 and W_2 be finite-dimensional vector spaces over a field K. Denote by n the dimension of V. Show that:

(a) $\operatorname{Hom}(V, W_1 \oplus W_2) \cong \operatorname{Hom}(V, W_1) \oplus \operatorname{Hom}(V, W_2)$

(b) $\operatorname{Hom}(V \oplus U, W_1) \cong \operatorname{Hom}(V, W_1) \oplus \operatorname{Hom}(U, W_1)$

(c) $\operatorname{Hom}(V, W) \cong \operatorname{Hom}(W^*, V^*)$

(d) The following map is an isomorphism
$$\begin{array}{ccc} \operatorname{Hom}(V, V) & \to & \operatorname{Hom}(V, V)^* \\ T & \mapsto & [S \mapsto \operatorname{tr}(S \circ T)], \end{array}$$
where $\operatorname{tr} \in \operatorname{Hom}(V, V)^*$ is the *trace map* defined by
$$\operatorname{tr}(T) = \operatorname{tr}([T]_{\mathcal{B}}^{\mathcal{B}}) = \sum_{i=1}^{n} ([T]_{\mathcal{B}}^{\mathcal{B}})_{ii},$$
for any basis $\mathcal{B}$ of V.

Remark. You may use this in the above exercise, but as a bonus you could first show that the map $T \in \operatorname{Hom}(V, V) \mapsto \operatorname{tr}(T)$ is independent of the choice of basis of V.

Solution:

(a) Consider the maps
$$\begin{array}{cccc} \Phi : & \operatorname{Hom}(V, W_1 \oplus W_2) & \to & \operatorname{Hom}(V, W_1) \oplus \operatorname{Hom}(V, W_2) \\ & \ell & \mapsto & (\mathrm{p}_{W_1} \circ \ell, \mathrm{p}_{W_2} \circ \ell)) \end{array}$$

and

$$\begin{array}{rccc} \Psi: & \mathrm{Hom}(V,W_1)\oplus \mathrm{Hom}(V,W_2) & \to & \mathrm{Hom}(V,W_1\oplus W_2) \\ & (\ell_1,\ell_2) & \mapsto & [\ell : v\mapsto (\ell_1(v),\ell_2(v))] \end{array}$$

The map Φ is well-defined since the composition of linear maps is linear. You can check that Ψ is well-defined directly from the definitions of the operations in the product $W_1 \oplus W_2$.

Moreover, a direct computations show that

$$\Psi\circ\Phi = \mathrm{id}_{\mathrm{Hom}(V,W_1\oplus W_2)} \quad \text{and} \quad \Phi\circ\Psi = \mathrm{id}_{\mathrm{Hom}(V,W_1)\oplus \mathrm{Hom}(V,W_2)}\,.$$

Using the linearity of the projections, it is also a straightforward computation to show that Φ is linear. So, Φ is a linear map with inverse Ψ, therefore it is an isomorphism.

(b) Consider the maps

$$\begin{array}{rccc} \Phi: & \mathrm{Hom}(V\oplus U,W_1) & \to & \mathrm{Hom}(V,W_1)\oplus \mathrm{Hom}(U,W_1) \\ & \ell & \mapsto & (\ell\circ\iota_V,\ell\circ\iota_U), \end{array}$$

where $\iota_V : V \to V\oplus U$ and $\iota_U : U \to V\oplus U$ are the canonical embeddings and

$$\begin{array}{rccc} \Psi: & \mathrm{Hom}(V,W_1)\oplus \mathrm{Hom}(U,W_1) & \to & \mathrm{Hom}(V\oplus U,W_1) \\ & (\ell_1,\ell_2) & \mapsto & [\ell : (v,u)\mapsto (\ell_1(v),\ell_2(u))] \end{array}$$

The fact that Φ and Ψ are well-defined follows from the similar reasons as in (a). As above, a straightforward computation shows that they are each other's inverse. Additionally, letting $\ell,\ell'\in \mathrm{Hom}(V\oplus U,W_1)$, we have

$$\begin{aligned} \Phi:\ell+\alpha\ell' \mapsto & ((\ell+\alpha\ell')\circ\iota_V,(\ell+\alpha\ell')\circ\iota_U) \\ =&(\ell\circ\iota_V+\alpha(\ell'\circ\iota_V),\ell\circ\iota_U+\alpha(\ell'\circ\iota_U)) \\ =&(\ell\circ\iota_V,\ell\circ\iota_U)+\alpha(\ell'\circ\iota_V,\ell'\circ\iota_U) \\ =&\Phi(\ell)+\alpha\Phi(\ell'). \end{aligned}$$

So, Φ is a linear map with inverse Ψ, therefore it is an isomorphism.

(c) Consider the map

$$\begin{array}{rccc} \Phi: & \mathrm{Hom}(V,W) & \to & \mathrm{Hom}(W^*,V^*) \\ & T & \mapsto & [\Phi(T):\ell\mapsto \ell\circ T,\ \forall \ell\in W^*] \end{array}$$

We first check that Φ is well-defined. Let $T\in \mathrm{Hom}(V,W)$, $\ell,\ell'\in W^*$, and $\alpha\in K$. Then,

$$\begin{aligned} \Phi(T)(\ell+\alpha\ell') &= (\ell+\alpha\ell')\circ T \\ &= \ell\circ T+\alpha(\ell'\circ T) \\ &= \Phi(T)(\ell)+\alpha\Phi(T)(\ell'). \end{aligned}$$

So, $\Phi(T)$ is indeed an element of $\mathrm{Hom}(W^*, V^*)$ for all $T \in \mathrm{Hom}(V, W)$.

Let us now show that Φ is injective. Assume that $\mathrm{Im}(T) \neq \{0\}$ and let $w \in \mathrm{Im}(T) \setminus \{0\}$. Extend $\{w\}$ to a basis $\mathcal{B}$ of W. Now define a map ℓ on $\mathcal{B}$ by setting

$$\ell(w) = 1 \quad \text{and} \quad \forall w' \in \mathcal{B} \setminus \{w\} : \ell(w') = 0$$

and extend it linearly to a functional of W. Then $\mathrm{Im}(T) \nsubseteq \ker(\ell)$ implies that $\Phi(T)(\ell)$ does not vanish on the whole of V and therefore that $\Phi(T)$ is not the 0 map. This shows that Φ is injective.

Now, let $\eta \in \mathrm{Hom}(W^*, V^*)$, let $\{e_i\}_{i=1}^{\dim(W)}$ be a basis of W, and define a map $T : V \to W$ by letting

$$T(v) := \sum_{i=1}^{\dim(W)} \eta(e_i^*)(v) e_i.$$

Since the sum is finite and since $\eta(e_i^*)$ is linear for all i, $T \in \mathrm{Hom}(V, W)$. Additionally, for any $\ell \in W^*$, for any $v \in V$

$$\begin{aligned}
\Phi(T)(\ell)(v) = \ell \circ T(v) &= \sum_{i=1}^{\dim(W)} \eta(e_i^*)(v)\ell(e_i) \\
&= \sum_{i=1}^{\dim(W)} \eta(\ell(e_i)e_i^*)(v) \\
&= \eta\left(\sum_{i=1}^{\dim(W)} \ell(e_i)e_i^*\right)(v) \\
&= \eta(\ell)(v),
\end{aligned}$$

since $\sum_{i=1}^{\dim(W)} \ell(e_i)e_i^* = \ell$. This shows that Φ is surjective.

(d) Let us first show that the trace of an application is well-defined. We will use

Lemma 1. *For any $A, B \in M_{n\times n}(K)$, we have*

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Beweis. This follows from a straight-forward computation. Denote by a_{ij}, respectively b_{ij}, the entries of A, respectively of B. Then the diagonal entries of the product AB are given by

$$(AB)_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki}, \quad 1 \leqslant i \leqslant n.$$

So,

$$\begin{aligned}\operatorname{tr}(AB) &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}a_{ik} \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \operatorname{tr}(BA).\end{aligned}$$

□

The independence of the choice of basis now follows. Observe that if $\mathcal{B}, \mathcal{C}$ are arbitrary bases of V and $T \in \operatorname{Hom}(V, V)$ we have

$$\begin{aligned}\operatorname{tr}([T]_{\mathcal{C}}^{\mathcal{C}}) &= \operatorname{tr}([\operatorname{id}_V]_{\mathcal{C}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}[\operatorname{id}_V]_{\mathcal{B}}^{\mathcal{C}}) \\ &= \operatorname{tr}([\operatorname{id}_V]_{\mathcal{C}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}([\operatorname{id}_V]_{\mathcal{C}}^{\mathcal{B}})^{-1}) \\ &= \operatorname{tr}([\operatorname{id}_V]_{\mathcal{C}}^{\mathcal{B}}([\operatorname{id}_V]_{\mathcal{C}}^{\mathcal{B}})^{-1}[T]_{\mathcal{B}}^{\mathcal{B}}) \\ &= \operatorname{tr}([T]_{\mathcal{B}}^{\mathcal{B}}),\end{aligned}$$

where the second-to-last equality follows from Lemma 1.
We now show that the map

$$\begin{array}{ccc}\operatorname{Hom}(V, V) & \to & \operatorname{Hom}(V, V)^* \\ T & \mapsto & [S \mapsto \operatorname{tr}(S \circ T)],\end{array}$$

is well-defined. This means showing that the map

$$\begin{array}{cccc}\operatorname{tr}_T : & \operatorname{Hom}(V, V) & \to & K \\ & S & \mapsto & \operatorname{tr}(S \circ T)\end{array}$$

is linear. We will need the following result:

Lemma 2. *The map* $\operatorname{tr} : M_{n\times n}(K) \to K$ *is linear. Namely, for any* $A, B \in M_{n\times n}(K)$*, and any* $\alpha \in K$*, the following hold:*

$$\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B) \quad \textit{and} \quad \operatorname{tr}(\alpha A) = \alpha \operatorname{tr}(A).$$

Beweis. Denote $A = (a_{ij})$ and $B = (b_{ij}) \in M_{n\times n}(K)$. We have

$$\begin{aligned}\operatorname{tr}(A + B) &= \sum_{k=1}^{n} (A + B)_{kk} \\ &= \sum_{k=1}^{n} a_{kk} + b_{kk} \\ &= \sum_{k=1}^{n} a_{kk} + \sum_{k=1}^{n} b_{kk} = \operatorname{tr}(A) + \operatorname{tr}(B).\end{aligned}$$

We now consider αA. We have,

$$\begin{aligned}\operatorname{tr}(\alpha A) &= \sum_{k=1}^{n} (\alpha A)_{kk} \\ &= \sum_{k=1}^{n} \alpha a_{kk} \\ &= \alpha \sum_{k=1}^{n} a_{kk} = \alpha \operatorname{tr}(A).\end{aligned}$$

□

Linearity of tr_T. Let $S, S' \in \operatorname{Hom}(V,V)$ and $\alpha \in K$, then

$$\begin{aligned}\operatorname{tr}_T(S + \alpha S') &= \operatorname{tr}((S + \alpha S') \circ T) \\ &= \operatorname{tr}(S \circ T + \alpha(S' \circ T)) \\ &= \operatorname{tr}([S \circ T + \alpha(S' \circ T)]_{\mathcal{B}}^{\mathcal{B}}) \\ &= \operatorname{tr}([S]_{\mathcal{B}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}} + \alpha [S']_{\mathcal{B}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}) \\ &= \operatorname{tr}([S]_{\mathcal{B}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}) + \alpha \operatorname{tr}([S']_{\mathcal{B}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}) \\ &= \operatorname{tr}_T(S) + \alpha \operatorname{tr}_T(S').\end{aligned}$$

We deduce that $\operatorname{tr}_T \in \operatorname{Hom}(V,V)^*$.

Injectivity. We now show that the above map is injective. Fix T and assume that tr_T is the 0 map. Then for all $S \in \operatorname{Hom}(V,V)$,

$$0 = \operatorname{tr}([S]_{\mathcal{B}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}) = \sum_{i=1}^{n} \sum_{k=1}^{n} s_{ik} t_{ki},$$

where we used the notation $T = (t_{ij})$ and $S = (s_{ij})$. Fix $(m,n) \in \{1, \dots, n\}^2$. Define $S_{m,n} = (\tilde{s}_{ij}) \in \operatorname{Hom}(V,V)$ such that all the entries of $[S_{m,n}]_{\mathcal{B}}^{\mathcal{B}}$ vanish except for $\tilde{s}_{mn} = 1$. We deduce that

$$0 = \operatorname{tr}([S_{m,n}]_{\mathcal{B}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}) = t_{nm}.$$

Since (m,n) was arbitrary, this shows that $[T]_{\mathcal{B}}^{\mathcal{B}} = 0$ and therefore that T is the 0 map.

To conclude, remember that V is finite -dimensional, which implies that $\operatorname{Hom}(V,V)$ is too. It follows that $\dim(\operatorname{Hom}(V,V)) = \dim(\operatorname{Hom}(V,V)^*)$ and hence, since the given map is injective, it is an isomorphism.

Single Choice. In each exercise, exactly one answer is correct.

1. Let $\dim V = 4$. Then there exists $\varphi \in V^*$ with $\dim \operatorname{Ker} \varphi = 2$.

   (a) Correct

   ✓ False

   Solution: $\varphi \in V^*$ means that $\varphi \in \operatorname{Hom}_K(V, K)$. If now $\dim \operatorname{Ker} \varphi = 2$ was true, the dimension formula

   $$\dim V = \dim \operatorname{Im} \varphi + \dim \operatorname{Ker} \varphi$$

   would yield $\dim \operatorname{Im} \varphi = 2$. This contradicts $\dim \operatorname{Im} \varphi \leqslant \dim K = 1$.

2. Every finite dimensional vector space is the dual of another finite dimensional vecor space.

   ✓ Correct

   (a) False

3. The set of all invertible $n \times n$-matrices is...

   ✓ not a real linear subspace of $M_n(\mathbb{R})$

   (a) a real linear subspace of $M_n(\mathbb{R})$

   Solution: The set is not closed under addition. Indeed, we have

   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   and the zero matrix is not invertible.

4. Let $f : V \to W$ be an arbitrary homomorphism between two K-vector spaces. Which of the following five assertions is not equivalent to the others?

   (a) f is injective.

   (b) The dual map $f^* : W^* \to V^*$ is surjective.

   (c) The zero element of V is the only element mapped to the zero element of W.

   ✓ There exists a Homomorphism $g : W \to V$ with $f \circ g = \mathrm{id}_W$.

   (d) For every $v \in V \backslash \{0\}$ there exists $\ell \in W^*$ with $\ell(f(v)) \neq 0$.

   (e) All five assertions are equivalent.

Solution: Assertion (d) is equivalent to f being surjective, but not equivalent to injectivity and hence not to (a). Assertion (a) is equivalent to $\mathrm{Kern}(f) = 0$ and thus to (c). As the exercise is correctly stated, (d) must be the correct answer. Indeed, (a) beeing equivalent to (b) was proved in exercise 17 of the repetition exercise sheet. Moreover, an element $w \in W$ is not equal to zero if and only if there exists $\ell \in W^*$ with $\ell(w) \neq 0$. Hence (e) is equivalent to $\forall v \in V \backslash \{0\} : f(v) \neq 0$, and hence equivalent to (a).

Multiple Choice Fragen.

1. For what value of parameter x is matrix $A = \left(\begin{smallmatrix} 1 & x & 1 \\ 3 & 3 & x \\ 0 & 3 & 1 \end{smallmatrix}\right)$ not invertible?

   (a) 0

   (b) 1

   ✓ 2

   (c) 3

   (d) 4