Question 1 (10 points)

Let $\mathcal{A}$ be an algebra of sets on a set X and $\lambda : \mathcal{A} \to [0, +\infty]$ a pre-measure on $\mathcal{A}$. We define the mapping $\mu : \mathcal{P}(X) \to [0, +\infty]$ as

$$\mu(E) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) \,\middle|\, A_1, A_2, \ldots \in \mathcal{A}, E \subseteq \bigcup_{k=1}^{\infty} A_k \right\}$$

for any set $E \subseteq X$. One can then show that μ is a measure.

1.Q1. [2 points] State Carathéodory's measurability criterion with respect to μ.

Solution: Carathéodory's criterion states that a set $A \subseteq X$ is measurable if and only if for any set $E \subseteq X$,

$$\mu(E) = \mu(E \setminus A) + \mu(E \cap A)$$

holds.

1.Q2. [4 points] Show that for every $A \in \mathcal{A}$, it holds that $\lambda(A) = \mu(A)$.

Solution: Since $\{A, \varnothing, \varnothing, \ldots\}$ is a covering of A by sets in $\mathcal{A}$ and $\lambda(\varnothing) = 0$, it clearly holds that $\mu(A) \leq \lambda(A)$.

For the reverse inequality, let $A_1, A_2, \ldots \in \mathcal{A}$ be such that $A \subseteq \bigcup_{k=1}^{\infty} A_i$ and define the sets $B_k := A_k \cap A \setminus (A_1 \cup \cdots \cup A_{k-1})$. Since $A \in \mathcal{A}$, by the properties of an algebra we see that $B_k \in \mathcal{A}$. Moreover they are pairwise disjoint by construction and satisfy

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A \cap A_k \setminus (A_1 \cup \cdots \cup A_{k-1}) = \bigcup_{k=1}^{\infty} A \cap A_k = A \cap \bigcup_{k=1}^{\infty} A_k = A.$$

Therefore $\lambda(A) = \sum_{k=1}^{\infty} \lambda(B_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$. Taking the infimum over all such collections we deduce that $\lambda(A) \leq \mu(A)$.

1.Q3. [4 points] Prove that every set $A \in \mathcal{A}$ is μ-measurable.

Solution: Let $E \in \mathcal{P}(X)$ be arbitrary. The inequality $\mu(E) \leq \mu(E \setminus A) + \mu(E \cap A)$ follows from the subadditivity of the measure μ.

For the opposite one, let $A_1, A_2, \ldots \in \mathcal{A}$ cover E and observe that the collections $\{A_i \setminus A\}$ and $\{A_i \cap A\}$ are covers of $E \setminus A$ and $E \cap A$ (respectively) also by sets in $\mathcal{A}$. Then, using the additivity of λ,

$$\mu(E \setminus A) + \mu(E \cap A) \leq \sum_{k=1}^{\infty} \lambda(A_k \setminus A) + \lambda(A_k \cap A) = \sum_{k=1}^{\infty} \lambda(A_k).$$

We now finish by taking the infimum over all such collections $\{A_i\}$.

Question 2 (14 points)

2.Q1. [2 points] Let (Ω, μ) be a measure space. For $1 \leq p < +\infty$, define the space $L^p(\Omega, \mu)$.

Solution: For $1 \leq p < +\infty$, we define $L^p(\Omega, \mu)$ to be the space of measurable functions $f : \Omega \to \overline{\mathbb{R}}$ such that

$$\|f\|^p_{L^p(\Omega,\mu)} := \int_\Omega |f|^p \, d\mu < +\infty,$$

up to almost everywhere equivalence.

Fix $1 \leq p < +\infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\Omega, \mu)$.

2.Q2. [4 points] Show that there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \|f_{n_k} - f_{n_{k+1}}\|_{L^p(\Omega,\mu)} < +\infty.$$

Solution: We take inducively n_k to be bigger than n_{k-1} and to satisfy that $\forall n \geq n_k$, $\|f_n - f_{n_k}\|_{L^p} \leq 2^{-k}$. This is possible thanks to the fact that $\{f_n\}$ is a Cauchy sequence in L^p. The desired sum clearly converges with this choice.

2.Q3. [4 points] Show that the function $g(x) := \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ is in $L^p(\Omega, \mu)$ and is finite μ-almost everywhere.

Solution: Let $g_K(x) := \sum_{k=1}^{K} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ and observe that $g_K \nearrow g$ monotonically almost everywhere. On the other hand, by the Minkowski inequality,

$$\|g_K\|_{L^p} \leq \sum_{k=1}^{K} \|f_{n_{k+1}}(x) - f_{n_k}(x)\|_{L^p} \leq C,$$

where C is the value of the finite sum in part (a). Observe that also $g_K^p \nearrow g^p$ monotonically almost everywhere, thus by Beppo Levi's theorem,

$$\int_\Omega g^p = \lim_{k\to\infty} \int_\Omega g_k^p = \lim_{k\to\infty} \|g_k\|^p_{L^p} \leq C^p < +\infty.$$

Hence g is in L^p and in particular finite almost everywhere.

2.Q4. [4 points] Prove that there exists a function $f \in L^p(\Omega, \mu)$ such that $\|f_{n_k} - f\|_{L^p(\Omega,\mu)} \to 0$ as $k \to \infty$ and deduce that $L^p(\Omega, \mu)$ is complete.

Solution: We have seen that for μ-almost every $x \in \Omega$ the sum $\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$ converges. Since an absolutely converging series of real numbers is converging, this means that the sequence $f_{n_K}(x) = f_{n_1}(x) + \sum_{k=1}^{K-1} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$ converges for almost every x to a value that we call $f(x)$. As the limit of measurable functions f is measurable. Moreover,

$$|f| \leq |f_{n_1}| + g \in L^p(\Omega, \mu)$$

and by applying dominated convergence to the functions $|f - f_{n_K}|^p \leq (|f| + |f_{n_K}|)^p \leq (|f| + g + |f_{n_1}|)^p \in L^1$ we deduce that

$$\lim_{K \to \infty} \|f - f_{n_K}\|_{L^p}^p = 0.$$

Finally, since the whole sequence $\{f_n\}$ is Cauchy and a subsequence of it converges, all the sequence must converge as well. Thus completeness is proven.

Question 3 (6 points)

Compute the following limits:

3.Q1. [3 points]

$$\lim_{m\to\infty}\int_0^\infty \frac{1}{1+x^m}\,dx$$

Solution: Observe that for $0 \le x < 1$, $x^m \to 0$ as $m \to \infty$, whereas for $x > 1$, $x^m \xrightarrow{m\to\infty} +\infty$. Hence, if we define $f_m(x)$ to be the integrand, $f_m \to \chi_{[0,1]}$ almost everywhere. In order to apply Lebesgue's dominated convergence theorem, we need to find a dominating function. For $x \le 1$, $f_m(x) \le 1$. On the other hand, for $x > 1$ and $m \ge 2$, observe that $x^2 \le x^m$, which implies that $f_m(x) \le \frac{1}{1+x^2}$. Thus the function $g(x) = \chi_{[0,1]}(x) + \frac{1}{1+x^2}$ dominates $\{f_m\}$ and is summable, hence we can pass to the limit

$$\lim_{m\to\infty}\int_0^\infty f_m(x)\,dx = \int_0^\infty \chi_{[0,1]}(x)\,dx = 1.$$

3.Q2. [3 points]

$$\lim_{m\to\infty}\int_0^1 \sum_{k=0}^m \frac{x^k}{k!}\,dx$$

Solution: Since the summands are positive, the sum inside the integral is a monotone function of m for each x. Therefore we can apply Beppo Levi's monotone convergence theorem and deduce

$$\lim_{m\to\infty}\int_0^1 \sum_{k=0}^m \frac{x^k}{k!}\,dx = \int_0^1 \sum_{k=0}^\infty \frac{x^k}{k!}\,dx = \int_0^1 e^x\,dx = e^1 - e^0 = e - 1.$$

Question 4 (8 points)

Let H be a *complex* vector space and consider a *function* $\langle\cdot,\cdot\rangle\colon H\times H\to\mathbb{C}$.

4.Q1. [2 points] Define what it means for the pair $(H,\langle\cdot,\cdot\rangle)$ to be a complex Hilbert space.

Solution: To start, $(H,\langle\cdot,\cdot\rangle)$ must be an inner product space. That is, the map $\langle\cdot,\cdot\rangle$ must be a positive-definite, conjugate-symmetric sesquilinear form, namely for all $u,v,w\in H$ and for all $\lambda\in\mathbb{C}$ we must have

$$\langle\lambda v,w\rangle=\lambda\langle v,w\rangle,\qquad\langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle,\qquad\overline{\langle v,w\rangle}=\langle w,v\rangle$$

and $\langle v,v\rangle>0$ unless $v=0$. Finally, the normed vector space $(H,\|\cdot\|)$ must be complete (i.e., all Cauchy sequences must have a limit), where the norm is defined through the scalar product as $\|v\|:=\sqrt{\langle v,v\rangle}$.

From now on assume that $(H,\langle\cdot,\cdot\rangle)$ is indeed a Hilbert space.

4.Q2. [3 points] Let $V\subset H$ be a vector subspace. Providing all the necessary assumptions, state the projection theorem on V. More precisely, define the closest-point projection operator $\pi_V\colon H\to V$ and characterize $\pi_V(v)$ (the projection of a point v) by a suitable orthogonality condition. No proofs are required.

Solution: Assume that $V\subset H$ is a closed linear subspace. Then for any $v\in H$ there exists a unique $\pi_V(v)\in V$ such that

$$\|v-\pi_V(v)\|=\min_{w\in V}\|v-w\|.$$

Moreover $\pi_V(x)$ is characterized by the following orthogonality property:

$$\langle v-\pi_V(v),w\rangle=0\quad\text{for any } w\in V.$$

4.Q3. [3 points] Assume $H:=L^2(\mathbb{R})$ with the standard L^2 scalar product and let V be the subspace of all odd functions in $L^2(\mathbb{R})$. After checking the necessary assumptions, prove that

$$\pi_V(f)(x)=\frac{f(x)-f(-x)}{2}.$$

Solution: First observe that the subspace of odd functions in $L^2(\mathbb{R})$ is linear and closed. Linearity is obvious; to check closedness let $(f_n)_{n\in\mathbb{N}}$ be a sequence of odd functions in $L^2(\mathbb{R})$ converging in $L^2(\mathbb{R})$ to a function f. Then, up to a subsequence, $(f_n)_{n\in\mathbb{N}}$ converges a.e. to f. It follows that $f_n(x)+f_n(-x)$ converges a.e. to $f(x)+f(-x)$, so that for a.e.

$x \in \mathbb{R}$

$$f(x) + f(-x) = \lim_{n\to\infty} f_n(x) + f_n(-x) = 0,$$

i.e. f is odd.
From the Projection Theorem it follows that there exists a projection π_{odd} to the subspace of odd functions. To check that π_{odd} has the form given in the exercise we first observe that for any $f \in L^2(\mathbb{R})$, $\pi_V(f)(x) = \dfrac{f(x) - f(-x)}{2}$ is odd. Finally we check that π_V satisfies the characterizing orthogonality condition: for any $g \in L^2(\mathbb{R})$ odd we have

$$\langle f - \pi_V(f), g\rangle = \int_{\mathbb{R}} \frac{f(x) + f(-x)}{2} g(x)dx = 0.$$

Here the integral is equal to zero since $\frac{f(x)+f(-x)}{2}$ is even and g is odd.

Question 5 (10 points)

5.Q1. [2 points] Compute the Fourier transform of $f(t) := \chi_{[-1/2,1/2]}(t), t \in \mathbb{R}$.

Solution:

$$\hat{f}(\xi) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(x)e^{-i\xi x}dx = (2\pi)^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\xi x}dx = (2\pi)^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\cos(\xi x) - i\sin(\xi x))\, dx$$

$$= (2\pi)^{-\frac{1}{2}} \frac{\sin(\xi x)}{\xi}\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin(\frac{\xi}{2})}{\xi}.$$

5.Q2. [3 points] Given u, v in $L^1(\mathbb{R})$, express $\mathcal{F}(u*v)$ in terms of $\hat{u}$ and $\hat{v}$. Prove rigorously your formula and specify whether $\mathcal{F}(u*v)$ is computed in the L^1 or in the L^2 sense.

Solution: Observe that $u * v \in L^1$ by Young's inequality. Thus we can compute

$$\begin{aligned}
\mathcal{F}(u*v)(\xi) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} u(x-y)v(y)dy \right) e^{-i\xi x}dx \\
&= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} u(x-y)e^{-i(x-y)\xi}v(y)e^{-iy\xi}dx \right) dy \\
&= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} u(z)e^{-iz\xi}v(y)e^{-iy\xi}dz \right) dy \\
&= \int_{\mathbb{R}} \hat{u}(\xi)v(y)e^{-iy\xi}dy = (2\pi)^{\frac{1}{2}}\hat{u}(\xi)\hat{v}(\xi),
\end{aligned}$$

where in the second step we used Fubini theorem, while in the third we used the substitution $z = x - y$ (for fixed y).

5.Q3. [3 points] Check that $g(t) := (f*f)(t)$ is equal to $(1-|t|)_+$ for all $t \in \mathbb{R}$ and compute $\hat{g}$.

Solution: First we check that $g(t) = (1 - |t|)_+$:

$$g(t) = \int_{\mathbb{R}} \chi_{[-\frac{1}{2},\frac{1}{2}]}(t-s)\chi_{[-\frac{1}{2},\frac{1}{2}]}(s)ds = \mathcal{L}^1\left(\left[t-\frac{1}{2}, t+\frac{1}{2}\right] \cap \left[-\frac{1}{2}, \frac{1}{2}\right]\right).$$

If $|t| > 1$, then the intersection $\left[t - \frac{1}{2}, t + \frac{1}{2}\right] \cap \left[-\frac{1}{2}, \frac{1}{2}\right]$ is empty and $g(t) = 0$.
If $t \in [0, 1]$, then $\left[t - \frac{1}{2}, t + \frac{1}{2}\right] \cap \left[-\frac{1}{2}, \frac{1}{2}\right] = \left[t - \frac{1}{2}, \frac{1}{2}\right]$ and $g(t) = \frac{1}{2} - (t - \frac{1}{2}) = 1 - t$.
If $t \in [-1, 0]$, then $\left[t - \frac{1}{2}, t + \frac{1}{2}\right] \cap \left[-\frac{1}{2}, \frac{1}{2}\right] = \left[-\frac{1}{2}, t + \frac{1}{2}\right]$ and $g(t) = t + \frac{1}{2} - \left(-\frac{1}{2}\right) = 1 + t$.
We conclude that $g(t) = (1 - |t|)_+$.

Next we compute $\hat{g}$. By the previous point we have

$$\hat{g}(\xi) = \mathcal{F}(f * f)(\xi) = (2\pi)^{\frac{1}{2}}(\hat{f}(\xi))^2 = (2\pi)^{\frac{1}{2}}\left(\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\sin(\frac{\xi}{2})}{\xi}\right)^2$$
$$= \frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\frac{\sin^2(\frac{\xi}{2})}{\xi^2} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1-\cos(\xi)}{\xi^2}.$$

5.Q4. [2 points] Does $\hat{g}$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$? Why?

Solution: From the previous point we see that $\xi^2\hat{g}(\xi)$ is periodic (and non-zero), therefore it is not decaying and cannot be in $\mathcal{S}(\mathbb{R})$. Alternatively one sees that $g \notin \mathcal{S}(\mathbb{R})$ as it is not smooth. Since the Fourier transform is a bijection from $\mathcal{S}(\mathbb{R})$ to itself, $\hat{g}$ cannot be in $\mathcal{S}(\mathbb{R})$.

Question 6 (12 points)

Consider the heat-type PDE

$$\partial_t u = \cos(t)\partial_{xx} u \text{ in } (0,T)\times\mathbb{R}, \qquad u(0^+,x) = f(x) \text{ for all } x\in\mathbb{R}, \tag{P}$$

where

- $T > 0$ is a given "final time",
- $u(t,x)$ is assumed to be real-valued and 2π-periodic in the x variable, that is $u(t,x) = u(t,x+2\pi)$ for all $t\in(0,T)$ and $x\in\mathbb{R}$, and
- $f(x)$ is a given initial condition which is also 2π-periodic.

Complete the following tasks:

6.Q1. [3 points] Assuming you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$, construct a formal solution w of (P) as a Fourier series in the x variable with t-dependent coefficients.

Solution: We write

$$w(t,x) = \sum_{k\in\mathbb{Z}} w_k(t)e^{ikx},$$

and from (P) we find for all $k\in\mathbb{Z}$ the ODEs

$$w_k'(t) + k^2\cos(t)w_k(t) = 0 \text{ in } (0,T), \quad w_k(0) = c_k(f).$$

Solving we find

$$w_k(t) = c_k(f)e^{-k^2\sin(t)} \text{ for all } t\in(0,T).$$

Hence the formal solution

$$w(t,x) := \sum_{k\in\mathbb{Z}} c_k(f)e^{ikx-k^2\sin(t)} \qquad \text{for all } (t,x)\in(0,T)\times\mathbb{R}.$$

6.Q2. [3 points] Check that, if $\int_{-\pi}^{\pi}|f|^2<\infty$ and $T<\pi$, then $w\colon(0,T)\times\mathbb{R}\to\mathbb{R}$ is indeed a well-defined continuous function.

Solution: By Parseval we know

$$\sum_{k\in\mathbb{Z}}|c_k(f)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2 < \infty.$$

Let us first show that the series defining w is absolutely convergent uniformly in compact subsets of $(0,T)\times\mathbb{R}$ as long as $T<\pi$. In fact if we fix any $\delta>0$ we have

$$\begin{aligned}\sum_{|k|\geq m}\sup_{(\delta,\pi-\delta)\times\mathbb{R}}|c_k(f)e^{ikx-k^2\sin(t)}| &= \sum_{|k|\geq m}|c_k(f)|\sup_{t\in(\delta,\pi-\delta)}|e^{-k^2\sin(t)}|\\ &\leq \|\{c_k(f)\}\|_{\ell^2}\Big(\sum_{|k|\geq m}e^{-2k^2\sin\delta}\Big)^{1/2}\xrightarrow{m\to\infty}0\end{aligned}$$

and the series is convergent (for example by the ratio test or any other Analysis I criterion). We used crucially that in $(0,\pi)$ the sine function is positive, so

$$\sup_{t\in(\delta,\pi-\delta)}|e^{-k^2\sin(t)}| = e^{-k^2\inf_{(\delta,\pi-\delta)}\sin} = e^{-k^2\sin\delta}.$$

We proved that $w\colon(0,T)\times\mathbb{R}\to\mathbb{R}$ is well-defined and continuous.

6.Q3. [3 points] Show that also the initial condition is met in the sense that

$$\lim_{t\downarrow 0}\|w(t,\cdot)-f\|_{L^2(-\pi,\pi)}=0.$$

Solution: For each fixed $t>0$ the kth Fourier coefficient of the 2π-periodic function $w(t,\cdot)$ is indeed given by $w_k(t)$. Hence by Parseval's identity we have

$$\frac{1}{2\pi}\|f-w(t,\cdot)\|^2_{L^2(-\pi,\pi)}=\sum_{k\in\mathbb{Z}}|c_k(f)-w_k(t)|^2=\sum_{k\in\mathbb{Z}}|c_k(f)|^2(1-e^{-k^2\sin(t)})^2.$$

We pass to the limit $t\downarrow 0$ using the dominated convergence theorem in $L^1(\mathbb{Z},\#,\mathcal{P}(\mathbb{Z}))$ and find

$$\lim_{t\downarrow 0}\|f-w(t,\cdot)\|^2_{L^2(-\pi,\pi)}=0.$$

The domination is given by

$$|c_k(f)|^2\underbrace{(1-e^{-k^2\sin(t)})^2}_{\leq 1}\leq|c_k(f)|^2\in\ell^1 \text{ by the assumption } f\in L^2.$$

6.Q4. [3 points] Show that w is in fact of class C^2 (in both variables) and solves the equation

$$\partial_t w=\cos(t)\partial_{xx}w \text{ in } (0,T)\times\mathbb{R}.$$

Solution: Now we have to do the same for the derivatives up to the second order. We first compute them

$$
\begin{aligned}
|\partial_t\big(e^{ikx-k^2\sin(t)}\big)| &= |\cos(t)k^2 e^{ikx-k^2\sin(t)}| \leq k^2|e^{-k^2\sin(t)}| \\
|\partial_x\big(e^{ikx-k^2\sin(t)}\big)| &= |ike^{ikx-k^2\sin(t)}| \leq |k||e^{-k^2\sin(t)}| \\
|\partial_{xx}\big(e^{ikx-k^2\sin(t)}\big)| &= |k^2 e^{ikx-k^2\sin(t)}| \leq k^2|e^{-k^2\sin(t)}| \\
|\partial_{xt}\big(e^{ikx-k^2\sin(t)}\big)| &= |k^4\cos(t)e^{ikx-k^2\sin(t)}| \leq k^4|e^{-k^2\sin(t)}| \\
|\partial_{tt}\big(e^{ikx-k^2\sin(t)}\big)| &= |k^2(k^2\cos^2(t)+\sin(t))e^{ikx-k^2\sin(t)}| \leq 2k^4|e^{-k^2\sin(t)}|.
\end{aligned}
$$

Hence we find that, as long as $\alpha+\beta \leq 2$ with $\alpha, \beta \in \mathbb{N}$, then

$$
\sum_{k\in\mathbb{Z}} \sup_{(\delta,\pi-\delta)\times\mathbb{R}} |c_k(f)\partial_t^\alpha \partial_x^\beta (e^{ikx-k^2\sin(t)})| \leq \sum_{k\in\mathbb{Z}} 2|c_k(f)|k^4 \sup_{t\in(\delta,\pi-\delta)} |e^{-k^2\sin(t)}|.
$$

By Cauchy Schwarz we conclude exactly as above since for every $\delta > 0$ it holds

$$
\sum_{k\in\mathbb{Z}} k^8 e^{-2k^2\sin\delta} < \infty.
$$

This proves that w is of class C^2 and its derivatives can be computed termwise differentiating the series that defines w. Hence it is readily checked that w satisfies (P) since

$$
\partial_t w = \sum_{k\in\mathbb{Z}} -k^2\cos(t)w_k(t)e^{ikx}, \quad \partial_{xx} w = \sum_{k\in\mathbb{Z}} -k^2 w_k(t)e^{ikx},
$$

and so $\partial_t w = \cos(t)\partial_{xx} w$.