ANALYSIS IV - MOCK EXAM - 90 MIN

Problem 1. Let H be a *complex* vector space and consider a *function* $\langle\cdot,\cdot\rangle\colon H\times H\to\mathbb{C}$.

(a) Define what it means that the pair $(H,\langle\cdot,\cdot\rangle)$ is a complex Hilbert space.

From now on assume that $(H,\langle\cdot,\cdot\rangle)$ is in fact a complex Hilbert space.

(b) State the parallelogram law and the Cauchy-Schwarz inequality in $(H,\langle\cdot,\cdot\rangle)$.

(c) Show that the Cauchy-Schwarz inequality implies the triangular inequality for the norm $\|v\|:=\sqrt{\langle v,v\rangle}$.

(d) Consider $H:=L^2((0,1),\mathbb{R})$ with the standard L^2 scalar product and the set $K:=\{v\in H : v\geq 0 \text{ a.e.}\}$. Prove that K is a convex and closed subset of H and that

$$P_K(u)(x)=\max\{u(x),0\},$$

where $P_K\colon H\to K$ denotes the closest point projection.

Problem 2.

(a) Given $f\in C_c^1(\mathbb{R})$, state and prove the formula expressing $\mathcal{F}(f')$ in terms of $\mathcal{F}(f)$.

(b) Compute the Fourier transform of $f(t):=e^{-|t|}, t\in\mathbb{R}$.

(c) Consider $g\colon\mathbb{R}\to\mathbb{R}$, the only 2π-periodic function that agrees with f in $[-\pi,\pi]$. Show that the Fourier partial sums $S_N(g)$ converge to g uniformly in $[-\pi,\pi]$.

Problem 3. Consider the heat-type PDE

$$\text{(P)}\qquad \partial_t u=\frac{1}{1+t^2}u+\partial_{xx}u \quad \text{in } (0,\infty)\times\mathbb{R},\quad u(0^+,x)=f(x) \text{ for all } x\in\mathbb{R},$$

where

- $u(t,x)$ is assumed to be real-valued and 2π-periodic in the x variable, that is $u(t,x)=u(t,x+2\pi)$ for all $t>0$ and $x\in\mathbb{R}$,
- $f(x)$ is a given initial condition which is also 2π-periodic.

Complete the following tasks:

(a) Assuming you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ construct a formal solution w of (P) as a Fourier series in the x variable with t-dependent coefficients.

(b) Check that, if $\int_{-\pi}^{\pi}|f|^2<\infty$, then $w\colon(0,\infty)\times\mathbb{R}\to\mathbb{R}$ is well-defined, of class C^2 and solves the equation

$$\partial_t w=\frac{1}{1+t^2}w+\partial_{xx}w \text{ in } (0,\infty)\times\mathbb{R}.$$

(c) Show that the initial condition is met in the sense that

$$\lim_{t\downarrow 0}\|w(t,\cdot)-f\|_{L^2(-\pi,\pi)}=0.$$

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You can use the results seen in class if you clearly identify them (either you call them by their name or you state the assumptions and the conclusion).

You can also give for granted the following facts:

- The definition of vector space over $\mathbb{C}$.
- The Fourier transform in $\mathbb{R}^d$ (under suitable assumptions) is given by
$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x}\, dx.$$
- $C_c^1(\mathbb{R})$ denotes the vector space of continuously differentiable functions from $\mathbb{R}$ to $\mathbb{C}$ each of which vanish outside a sufficiently large interval.
- For a 2π-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ the kth fourier coefficient is given by
$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \qquad \text{for each } k \in \mathbb{Z}.$$
Under suitable assumption f can be expressed as limit of the Fourier partial sums
$$S_N f(x) = \sum_{|k|\le N} c_k(f) e^{ikx}.$$