ANALYSIS IV - MOCK EXAM - 90 MIN

Problem 1. Let H be a *complex* vector space and consider a *function* $\langle\cdot,\cdot\rangle\colon H\times H\to\mathbb{C}$.

(a) Define what it means that the pair $(H,\langle\cdot,\cdot\rangle)$ is a complex Hilbert space.

From now on assume that $(H,\langle\cdot,\cdot\rangle)$ is in fact a complex Hilbert space.

(b) State the parallelogram law and the Cauchy-Schwarz inequality in $(H,\langle\cdot,\cdot\rangle)$.

(c) Show that the Cauchy-Schwarz inequality implies the triangular inequality for the norm $\|v\|:=\sqrt{\langle v,v\rangle}$.

(d) Consider $H:=L^2((0,1),\mathbb{R})$ with the standard L^2 scalar product and the set $K:=\{v\in H: v\geq 0 \text{ a.e.}\}$. Prove that K is a convex and closed subset of H and that

$$P_K(u)(x)=\max\{u(x),0\},$$

where $P_K\colon H\to K$ denotes the closest point projection.

Problem 2.

(a) Given $f\in C^1_c(\mathbb{R})$, state and prove the formula expressing $\mathcal{F}(f')$ in terms of $\mathcal{F}(f)$.

(b) Compute the Fourier transform of $f(t):=e^{-|t|}, t\in\mathbb{R}$.

(c) Consider $g\colon\mathbb{R}\to\mathbb{R}$, the only 2π-periodic function that agrees with f in $[-\pi,\pi]$. Show that the Fourier partial sums $S_N(g)$ converge to g uniformly in $[-\pi,\pi]$.

Problem 3. Consider the heat-type PDE

$$\text{(P)}\qquad \partial_t u=\frac{1}{1+t^2}u+\partial_{xx}u \quad \text{in } (0,\infty)\times\mathbb{R},\quad u(0^+,x)=f(x) \text{ for all } x\in\mathbb{R},$$

where

- $u(t,x)$ is assumed to be real-valued and 2π-periodic in the x variable, that is $u(t,x)=u(t,x+2\pi)$ for all $t>0$ and $x\in\mathbb{R}$,
- $f(x)$ is a given initial condition which is also 2π-periodic.

Complete the following tasks:

(a) Assuming you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ construct a formal solution w of (P) as a Fourier series in the x variable with t-dependent coefficients.

(b) Check that, if $\int_{-\pi}^{\pi}|f|^2<\infty$, then $w\colon(0,\infty)\times\mathbb{R}\to\mathbb{R}$ is well-defined, of class C^2 and solves the equation

$$\partial_t w=\frac{1}{1+t^2}w+\partial_{xx}w \text{ in } (0,\infty)\times\mathbb{R}.$$

(c) Show that the initial condition is met in the sense that

$$\lim_{t\downarrow 0}\|w(t,\cdot)-f\|_{L^2(-\pi,\pi)}=0.$$

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You can use the results seen in class if you clearly identify them (either you call them by their name or you state unambiguously the assumptions and the conclusion).

You can also give for granted the following facts:

- The definition of vector space over $\mathbb{C}$.
- The Fourier transform in $\mathbb{R}^d$ (under suitable assumptions) is given by
$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x}\, dx.$$
- $C^1_c(\mathbb{R})$ denotes the vector space of continuously differentiable functions from $\mathbb{R}$ to $\mathbb{C}$ each of which vanish outside a sufficiently large interval.
- For a 2π-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ the kth fourier coefficient is given by
$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \qquad \text{for each } k \in \mathbb{Z}.$$
Under suitable assumption f can be expressed as a suitable limit of the Fourier partial sums
$$S_N f(x) = \sum_{|k|\le N} c_k(f) e^{ikx}.$$

Solutions.

(1a) H must be a complex vector space (whose definition we can give for granted). $\langle\cdot,\cdot\rangle$ must be a positive-definite, conjugate symmetric sesquilinear form. That is for all $x, y, z \in H$ and for all $\lambda \in \mathbb{C}$ we must have

$$\langle \lambda x, y\rangle = \lambda\overline{\langle y, x\rangle}, \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$

and $\langle x, x\rangle > 0$ unless $x = 0$.

Finally, the normed vector space $(H, \|\cdot\|)$ must be complete (i.e., all Cauchy sequences must have a limit), where the norm is defined trough the scalar product as $\|x\| := \sqrt{\langle x, x\rangle}$.

(1b) Parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in H$.

Cauchy-Schwarz: $|\langle x, y\rangle| \le \|x\|\|y\|$, for all $x, y \in H$.

(1c) Just take the square root of

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x, y\rangle)\\ &\le \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle|\\ &\le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2.\end{aligned}$$

In the first inequality we used that for any complex number $Re(z) \le |z|$ and in the second we used the C.S. inequality.

(1d) Clearly, if v_1, v_2 are non-negative and $t \in [0,1]$, then also $tv_1 + (1-t)v_2$ is non-negative, hence K is convex. We remark also that $K \neq \emptyset$

Let $\{v_j\} \subset K$ be a sequence with $v_j \to v_\infty$ in $L^2(0,1)$. Up to a subsequence, we can assume that the convergence holds also almost everywhere, now since each v_j was non-negative, so must be its pointwise a.e. limit.

Let us call $w \in L^2(0,1)$ the projection of u. Then w is uniquely determined by the conditions:

$$w \in K \text{ and } \langle u - w, v - w\rangle \le 0 \text{ for all } v \in K. \tag{0.1}$$

Now we set

$$u_+ := \max\{u, 0\} \text{ and } u_- := \max\{-u, 0\},$$

so that $u = u_+ - u_-$. We check that u_+ satisfies (0.1) and thus $w = u_+$.

With our notation we have $u = u_+ - u_-$ and so for any $v \ge 0$ we have

$$\langle u - u_+, u_+ - v\rangle = -\underbrace{\langle u_-, v\rangle}_{\ge 0} + \underbrace{\langle u_-, u_+\rangle}_{=0} \le 0.$$

The first sign is justified because both u_- and v are non-negative, so $\int u_- v \ge 0$. The second is due to the fact that $u_+ u_- \equiv 0$, since where one between $u_\pm$ is nonzero the other is zero.

(2a) The formula is

$$\mathcal{F}(f')(\xi) = i\xi\mathcal{F}(f)(\xi) \text{ for all } \xi \in \mathbb{R},$$

both sides are well-defined continuous functions since both f and f' are in $L^1(\mathbb{R})$ (being compactly supported), and we saw in class that

$$\mathcal{F}(L^1(\mathbb{R})) \subset C(\mathbb{R}).$$

In order to prove this we notice that $f(x)e^{-i\xi x}$ is in $C_c^1(\mathbb{R})$ so the integral of its derivative is zero, hence we find

$$0 = \int_{\mathbb{R}} \frac{d}{dx}\big(f(x)e^{-i\xi x}\big)\,dx = \int_{\mathbb{R}} f'(x)e^{-i\xi x}\,dx - i\xi\int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx$$

which proves the formula up to the dividing by the factor $\sqrt{2\pi}$.

(2b) Since e^{-t} is in L^1 we can use the integral formula for $\mathcal{F}$ and we have to compute

$$\begin{aligned}\sqrt{2\pi}\hat{f}(\xi) &= \int_0^\infty e^{-x-i\xi x}\,dx + \int_{-\infty}^0 e^{x-i\xi x}\,dx \\ &= \Big[-\frac{e^{-(1+i\xi)x}}{1+i\xi}\Big]_{x=0}^{\infty} + \Big[\frac{e^{(1-i\xi)x}}{1-i\xi}\Big]_{x=-\infty}^{0} \\ &= \frac{1}{1+i\xi} + \frac{1}{1-i\xi} = \frac{2}{1+\xi^2}\end{aligned}$$

(2c) g is a of class C^1 in $[-\pi,\pi]$ and is attains the same value at the extrema of the interval $x=\pm\pi$ (because it is even). Hence we saw in class that these two conditions guarantee uniform convergence of the Fourier partial sums to the original function.

The key point was that we could integrate by parts (thanks to the periodic condition no boundary term) and find $\{kc_k(g)\}\in\ell^2$, which entails $\{c_k(g)\}\in\ell^1$.

(3a) We write

$$w(t,x) = \sum_{k\in\mathbb{Z}} w_k(t)e^{-ikx},$$

deriving formally, we find

$$\partial_t w = \sum_{k\in\mathbb{Z}} w_k'(t)e^{-ikx}, \frac{1}{1+t^2}w = \sum_{k\in\mathbb{Z}} \frac{1}{1+t^2} w_k(t)e^{-ikx}$$

$$\text{and } \partial_{xx} w = \sum_{k\in\mathbb{Z}} -k^2 w_k(t)e^{-ikx}.$$

Moreover, we formally have

$$f(x) = w(0,x) = \sum_{k\in\mathbb{Z}} w_k(0)e^{-ikx},$$

so we set $w_(0) = c_k(f)$ for each k.

We get that for any $k\in\mathbb{Z}$, w_k must satisfy the Cauchy problem

$$\begin{cases} w_k' = \big(\frac{1}{1+t^2} - k^2\big)w_k, \\ w_k(0) = c_k(f). \end{cases}$$

Integrating we find

$$w_k(t) = c_k(f)e^{\arctan t - k^2 t}.$$

So the formal solution

$$w(t,x) = \sum_{k\in\mathbb{Z}} c_k(f)e^{\arctan t - k^2 t + ikx}.$$

(3b) We first check that the series defining w is totally convergent on compact subsets of $(0,\infty)\times\mathbb{R}$. To do so, it suffices to show that for each $\delta>0$ we have

$$\sum_{k\in\mathbb{Z}} \sup_{(\delta,1/\delta)\times\mathbb{R}} |w_k(t)e^{-ikx}| < \infty.$$

This is readily checked since

$$\begin{aligned}\sup_{(\delta,1/\delta)\times\mathbb{R}} |w_k(t)e^{-ikx}| &= |c_k(f)| \sup_{t\in(\delta,1/\delta)} |e^{\arctan t}e^{-k^2 t}| \\ &\le |c_k(f)| \sup_{t\in\mathbb{R}} e^{\arctan t} \sup_{t\in(\delta,1/\delta)} e^{-k^2 t} \le |c_k(f)|e^{\pi/2}e^{-k^2\delta}.\end{aligned}$$

To check that this is summable we have to use the assumption $f \in L^2$ which by Parseval implies

$$\sum_{k\in\mathbb{Z}} |c_k(f)|^2 = \int_{-\pi}^{\pi} |f|^2 < \infty,$$

and so by Cauchy Schwarz and the previous bound

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sup_{(\delta,1/\delta)\times\mathbb{R}} |w_k(t)e^{-ikx}| &\le \sum_{k\in\mathbb{Z}} |c_k(f)| e^{\pi/2} e^{-k^2\delta} \\ &\le \Big(\sum_{k\in\mathbb{Z}} |c_k(f)|^2\Big)^{1/2} \Big(\sum_{k\in\mathbb{Z}} e^{\pi-2k^2\delta}\Big)^{1/2} < \infty.\end{aligned}$$

This implies $w \in C((0,\infty)\times\mathbb{R})$.

In order to show that w is in fact C^2 (actually, it is C^∞) it suffices to show that for each $\delta > 0$ and p, q nonnegative integers with $p + q \le 2$ it holds

$$\sum_{k\in\mathbb{Z}} \sup_{(\delta,1/\delta)\times\mathbb{R}} |\partial_t^p \partial_x^q (w_k(t)e^{-ikx})| < \infty.$$

We compute all of them and show that they are bounded by a common converging series. Set $g(t) := e^{\arctan t}$, and notice that since it is smooth on the whole $\mathbb{R}$ we have

$$M_\delta := \max_{[\delta,1/\delta]} |g| + |g'| + |g''| < \infty.$$

Then for all $(t,x) \in (\delta, 1/\delta)\times\mathbb{R}$ we have

$$\begin{aligned}|g(t)e^{-k^2t-ikx}| &\le e^{-k^2\delta} M_\delta \\ |\partial_t(g(t)e^{-k^2t-ikx})| &\le \left|(g'(t) - k^2 g(t))e^{-k^2t}\right| \le M_\delta(1+k^2)e^{-k^2\delta} \\ |\partial_{tt}(g(t)e^{-k^2t-ikx})| &\le \left|(g''(t) - 2g'(t)k^2 + k^4 g(t))e^{-k^2t}\right| \le M_\delta(1+2k^4)e^{-k^2\delta} \\ |\partial_x(g(t)e^{-k^2t-ikx})| &\le M_\delta |k| e^{-k^2\delta} \\ |\partial_{xx}(g(t)e^{-k^2t-ikx})| &\le M_\delta k^2 e^{-k^2\delta}.\end{aligned}$$

So taking the worst case we have anyway

$$\sum_{k\in\mathbb{Z}} \sup_{(\delta,1/\delta)\times\mathbb{R}} |\partial_t^p \partial_x^q (w_k(t)e^{-ikx})| \le 4M_\delta \sum_{k\in\mathbb{Z}} |c_k(f)|(1+k^4)e^{-k^2\delta},$$

which is convergent by Cauchy-Schwarz as above.

This proves that $w \in C^2((0,\infty)\times\mathbb{R})$ and that the derivatives can be computed differntiating the series termwise. So the PDE is satisfied

$$\begin{aligned}\partial_t w(t,x) &= \partial_t\left(\sum_{k\in\mathbb{Z}} w_k(t)e^{ikx}\right) = \sum_{k\in\mathbb{Z}} w_k'(t)e^{ikx} = \sum_{k\in\mathbb{Z}} \left(\frac{1}{1+t^2} - k^2\right) w_k(t)e^{ikx} \\ &= \sum_{k\in\mathbb{Z}} \frac{1}{1+t^2} w_k(t)e^{ikx} - w_k(t)\partial_{xx}e^{ikx} = \frac{1}{1+t^2}\sum_{k\in\mathbb{Z}} w_k(t)e^{ikx} - \partial_{xx}\left(\sum_{k\in\mathbb{Z}} w_k(t)e^{ikx}\right) \\ &= \frac{1}{1+t^2} w(t,x) - \partial_{xx} w(t,x).\end{aligned}$$

Of course, so far, we did not prove anything concerning what happens for $t \downarrow 0$.

(3c) Again by applying Parseval's identity, we find

$$\begin{aligned}\lim_{t\downarrow 0}\|w(t,\cdot)-f\|^2_{L^2(-\pi,\pi)} &= \lim_{t\downarrow 0}\sum_{k\in\mathbb{Z}}\left|c_k(f)e^{\arctan t-k^2t}-c_k(f)\right|^2\\ &= \lim_{t\downarrow 0}\sum_{k\in\mathbb{Z}}|c_k(f)|^2\left|e^{\arctan t-k^2t}-1\right|^2\\ &\le \lim_{t\downarrow 0}\sum_{k\in\mathbb{Z}}|c_k(f)|^2\underbrace{\left|e^{-k^2t}-1\right|^2}_{\to 0\text{ for fixed }k}=0.\end{aligned}$$

The last passage holds by the Dominated Convergence Theorem. Indeed, the series on the right-hands side is the integral in $L^1(\mathbb{Z},\mathcal{P}(\mathbb{Z}),\#)$ of $\phi_t(k) := |c_k(f)|^2\left|e^{-k^2t}-1\right|^2$, which is dominated uniformly in t, since

$$|\phi_k(t)|\le 4|c_k(f)|^2\in L^1(\mathbb{Z},\mathcal{P}(\mathbb{Z}),\#)$$

and converges pointwise (i.e., for each k) to 0 as $t\to 0$.