These closed-answer questions cover some topics from previous classes that will be useful for this course. If you find some of them obscure you are encouraged to revise briefly the relative topic and/or to come to office hours.

1.1. Density (i.e., approximability) of certain functions in L^p spaces.

1. Is it true that continuous functions are dense in $L^1(0,1)$? And in $L^\infty(0,1)$?
2. Is it true that step functions are dense in $L^1(0,1)$? And in $L^2(\mathbb{R}^n)$?
3. Is it true that every function in $L^\infty(0,1)$ can be approximated by continuous functions with respect to the L^1-convergence?

The next three problems concern more directly what you have seen in class.

1.2. Continuity of operations. An inner product space $(V, \langle\cdot,\cdot\rangle)$ is also a metric space under the norm $|\cdot| := \sqrt{\langle\cdot,\cdot\rangle}$, hence it has a natural topology. Prove that $\langle\cdot,\cdot\rangle$ and the vector space operations $(\cdot, +)$ are continuous from $V \times V$ (resp. $V \times \mathbb{C}$, $V \times V$) endowed with the natural product topology, to $\mathbb{C}$ (resp. V, V).

1.3. Topology of normed spaces. Determine whether the following sets X are well-defined, open/close/none, and subspaces/convex/none.

1. In the normed space $(C([0,1], \|\cdot\|_{L^\infty}))$, the subset X of nowhere vanishing functions.
2. In the normed space $(C([0,1]), \|\cdot\|_{L^2})$, the subset X of nowhere vanishing functions.
3. In the normed space $(L^2(0,1), \|\cdot\|_{L^2})$, the subset $X = \{f : \int_0^1 f = 1\}$.
4. In the normed space $(L^2(0,1), \|\cdot\|_{L^2})$, the subset $X = \{f : f \geq 0 \text{ and } \int_0^1 \frac{2f}{1+f} \geq 1\}$.
5. In the normed space $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$, the subset $\{f : f(x) = f(-x) \text{ for a.e. } x \in \mathbb{R}\}$.

1.4. Quantitative Cauchy Schwarz. Let H be a real inner product space, prove the identity

$$|x||y| - x \cdot y = \frac{|x||y|}{2}\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2 \geq 0 \text{ for all } x, y, \in H.$$

1. Characterize the set $C \subset H \times H$ of pair of vectors that saturate the Cauchy-Schwarz inequality, i.e. $x \cdot y = |x||y|$. Plot C in the case $H = \mathbb{R}$.
2. If x, y are ϵ-close to saturate the Cauchy Schwarz inequality, that is

   $$(1-\epsilon)|x||y| \leq x \cdot y,$$

   then how close are x, y to the set C? Bound from above the number

   $$\inf_{(x',y') \in C} |x - x'|^2 + |y - y'|^2 =: dist^2((x,y), C).$$