Hints in the next page!

10.1. Closed answer questions.

1. If g_k are continuous and compactly supported functions in $\mathbb{R}^d$ such that $g_k \to g$ uniformly, is it true that g is necessarily continuous? Vanishes as $|x| \to \infty$? Has compact support?
2. Is there a function $f \in L^1(\mathbb{R}^2)$ such that $\hat{f}(\xi_1, \xi_2) = \frac{\sin(\xi_2)}{1+i\xi_1^2}$?
3. Let $\phi \in L^1(\mathbb{R}^d)$ and consider $\phi_t(x) := \phi(x)\mathbf{1}_{\{|\phi(x)|\geq t\}}$, for $t > 0$. Is it true that

$$\sup_{\xi \in \mathbb{R}^d} |\mathcal{F}(\phi_t)(\xi)| \to 0 \text{ as } t \to \infty?$$

4. Compute the Fourier transform of the indicator function of the interval $\mathbf{1}_{[-1,1]}(x)$, for $x \in \mathbb{R}$.
5. Given $f \in L^1(\mathbb{R}^d)$ explain how $f \star f$ is defined and why $(f \star f)(0)$ is not necessarily a well-defined number (an example suffices).

10.2. Heat equation for rough initial data. You are given $f \in L^2(-\pi, \pi)$, and consider the associated heat equation solution defined by

$$u(t,x) := \sum_{k \in \mathbb{Z}} c_k(f) e^{ikx - k^2 t}, \text{ for all } x \in \mathbb{R}, t > 0.$$

1. Show that $u \in C^\infty((0,\infty) \times \mathbb{R})$ and solves the heat equation

$$\partial_t u(t,x) = \partial_{xx} u(t,x), \qquad \text{for all } x \in \mathbb{R}, t > 0. \tag{1}$$

2. Show that u assumes the initial datum f in the following L^2 sense

$$\lim_{t \downarrow 0} \|u(t,\cdot) - f\|_{L^2(-\pi,\pi)} = 0. \tag{2}$$

3. Consider a function $v(t,x)$ defined in $(0,\infty) \times \mathbb{R}$ which is of class C^2 in space and 2π-periodic, and of class C^1 in time. If v satisfies equations (1) and (2), then $v = u$.

10.3. The wave equation.

Consider the evolution problem with periodic boundary conditions:

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = \lambda u & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \text{ where } \lambda \leq 0 \text{ is a given constant,} \\ u(t,x) = u(t, x + 2\pi) & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } f \in C^\infty(\mathbb{R}), 2\pi\text{-periodic,} \\ \partial_t u(0,x) = g(x) & \text{for some given } g \in C^\infty(\mathbb{R}), 2\pi\text{-periodic.} \end{cases}$$

1. Write the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, where the $\{u_k(t)\}$ depend on λ and the Fourier coefficients of f and g.
2. Show that the formal solution is in fact a true solution and is C^∞ in both variables.
3. Show that, if we just want our solution u to be $C^2(\mathbb{R}\times\mathbb{R})$, the assumptions on f and g can be relaxed to:
$$\sum_{k\in\mathbb{Z}} |k|^2|c_k(f)| + |k||c_k(g)| < +\infty.$$
4. Show that we found the only possible solution: if v is another solution of the problem above which is $C^2(\mathbb{R}\times\mathbb{R})$, then $u = v$.
5. Assume that $\lambda = 0$. Show that for each pair $\phi, \psi \in C^2_{per}$ the function $(x,t) \mapsto \phi(x-t) + \psi(x+t)$ solves the wave equation, explain why this is compatible with what you found in the previous points.
6. ($\star$) Does the wave equation have the "smoothing effect" for positive times?

Hints:

10.1.1. The vector space of continuous functions vanishing at infinity is *complete* in the L^∞ norm...

10.1.2. Recall what you proved in class concerning the Fourier transform of functions in L^1.

10.1.3. $\|\phi_t\|_{L^1} \to 0$ as $t \to \infty$... (why?)

10.1.4. Use the definition...

10.1.5. Either work by approximation, combining Young's inequality with the density of continuous compactly supported functions in $L^p(\mathbb{R}^d)$, either use the Fubini-Tonelli Theorem... for the counterexample use an $f \in L^1_{loc}(\mathbb{R}) \setminus L^2_{loc}(\mathbb{R})$...

10.2.1 Argue as in class, only use that this time you only know that $\sum_k |c_k(f)|^2 < \infty$ (instead of $\sum_k |c_k(f)| < \infty$!)

10.2.2. Use Parseval's identity and the dominated convergence Theorem in $L^2(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#)$...

10.2.3. Argue as in class...

10.3.1. Recall that λ has a sign... you will get the equation of the harmonic oscillator...

10.3.2. Recall that the Fourier coefficients of f and g decay faster than any power of $1/k$... So by the previous formula also the ones of u must decay...

10.3.3. If you check the computations of the previous point, the given decay on $c_k(f), c_k(g)$ is just enough to guarantee that $u \in C^2_{x,t}$...

10.3.4. Argue as in the case of the heat equation, showing that $u_k(t)$ and $v_k(t)$ obey the same ODE with the same initial conditions... hence they must agree for every time for which they are defined!

10.3.5. Show that the solution in Fourier series can indeed be written as $\phi(x-t) + \psi(x+t)$ for an appropriate pair of functions ϕ, ψ...