Hints in the next page!

11.1. Some Fourier transforms. Compute the following one dimensional Fourier transforms for

$$e^{ix-|x|^2}, \quad e^{-a|x|}, \quad e^{-ax}\sin(3x)\mathbf{1}_{(0,\infty)},$$

where $a \in \mathbb{C}, \operatorname{Re}(a) > 0$. Given $u, v \in \mathcal{S}(\mathbb{R})$ compute the Fourier transforms of

$$(x, y) \mapsto u(2x)v(y/2)$$

in terms of $\hat{u}, \hat{v}$.

11.2. Dominated convergence review. Motivate each of the following statements using the dominated convergence theorem in a suitable measure space, but pay attention: one of them is in fact *false*!

1. Given $f \in L^2(\mathbb{R}^d)$ it holds

$$\int_{\{|x|>R\}} f(x)^2 \sin(x_1)\, dx \to 0 \text{ as } R \to \infty$$

2. Given $f \in L^1(\mathbb{R}^d)$ it holds

$$\int_{\{|x|>R\}} \frac{\sqrt{1+f(x)^2}-1}{1+\hat{f}(x)}\, dx \to 0 \text{ as } R \to \infty$$

3. Let $\psi \in C^1_c(\mathbb{R}^d)$ such that $\psi(x) \equiv 1$ in a neighbourhood of $x = 0$. Then for each $f \in L^1(\mathbb{R}^d)$ it holds

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d} f(x)\psi(\epsilon x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx \quad \text{and} \quad \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} f(x)\partial_{x_j}\psi(\epsilon x)\, dx = 0.$$

4. Let $\{c_k\} \in \ell^1(\mathbb{N})$. The map $f(t) := \sum_{k\in\mathbb{N}} c_k e^{i\sin(k)t}$ is of class $C^\infty(\mathbb{R})$ and its derivatives are given by $f^{(m)}(t) = \sum_{k\in\mathbb{N}}(i\sin(k))^m c_k e^{i\sin(k)t}$.
5. Let $\{c_k\} \in \ell^2(\mathbb{N})$. The map $f(t) := \sum_{k\in\mathbb{N}} c_k e^{ikt^2}$ is of class $C^1(\mathbb{R})$ and its derivative is given by $f'(t) = 2it\sum_{k\in\mathbb{N}} kc_k e^{ikt^2}$.

11.3. Harmonic functions on the disk. In this problem we show the existence of the so-called harmonic extension in the interior of the disk of a sufficiently regular function f defined on the disk boundary.

Consider the second order differential operators in two variables (x_1, x_2):

$$\Delta := \partial_{11} + \partial_{22} \qquad \text{and} \qquad L := \partial_{11} + \frac{1}{x_1}\partial_1 + \frac{1}{x_1^2}\partial_{22}.$$

We say that a twice differentiable function $w(x_1, x_2)$ is harmonic $\Delta w = 0$ in its domain.

1. Given $u\colon \overline{D} \to \mathbb{R}$, where $D := \{(x, y) : x^2 + y^2 < 1\}$, consider the function
$$v(r, \theta) := u(r\cos\theta, r\sin\theta), \quad r \in [0, 1], \theta \in \mathbb{R}.^{1} \tag{1}$$
Using the chain rule, check that $(\Delta u)(r\cos\theta, r\sin\theta) = Lv(r, \theta)$ for all $r \in (0, 1)$ and $\theta \in \mathbb{R}$.

2. Given any regular function $F\colon \partial D \to \mathbb{R}$ consider its 2π periodic version $f\colon \mathbb{R} \to \mathbb{R}$ defined by
$$F(\cos\theta, \sin\theta) =: f(\theta), \quad \theta \in \mathbb{R}.$$
Show that we can find a solution of
$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \{0\}, \\ u = F & \text{on } \partial D, \end{cases}$$
solving instead
$$\begin{cases} \partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0 & \text{in } (0, 1] \times \mathbb{R}, \\ v(r, \theta + 2\pi) = v(r, \theta) & \text{in } (0, 1] \times \mathbb{R}, \\ v(1, \theta) = f(\theta) & \text{for all } \theta \in \mathbb{R}, \end{cases} \tag{2}$$
and then defining u trough (1).

3. Formally solve the system (2) by the Ansatz $v := \sum_{k\in\mathbb{Z}} u_k(r)e^{ik\theta}$. Explain why the $\{u_k(r)\}$ are not *uniquely* determined by the $\{c_k(f)\}$. Explain why they are unique if we further require that
$$\lim_{r\downarrow 0} u_k(r) = 0 \text{ for all } k \in \mathbb{Z}. \tag{3}$$

4. Let $v(r, \theta)$ be the Ansatz constructed in the previous point using the extra assumption (3). Show that v is of class C^∞ in the (r, θ) variables in $[0, 1) \times \mathbb{R}$, as soon as $f \in L^2$.

5. ($\star$) show that, as soon as $f \in L^2(-\pi, \pi)$, the v you constructed with the extra assumption (3), corresponds in fact to a u that is $C^\infty(D)$ in the whole open disk including the origin!). Furthermore this u meets the boundary condition in the sense that
$$\lim_{r\uparrow 1} \|u(r, \cdot) - f\|_{L^2(-\pi,\pi)} = 0.$$

[1] This is u in polar coordinates.

Hints:

11.1. The first are elementary integrals. Use the dilation properties for the last.

11.2.2. Recall what you known about the Fourier transform of an L^1 function.

11.3.1. Use the chain rule.

11.3.2. This is a simple verification.

11.3.3. You will find a second-order ODE, but only one initial condition at $r = 1$. Imposing that u vanishes at $r = 0$ acts as second condition.

11.3.4. You can show that when $r < 1$ the Fourier coefficients decay faster than any power.

11.3.5. You need to express $\partial_x u$ and $\partial_y u$ as a linear combination of $\partial_\theta, \partial_r$, in polar coordinates. Then check continuity at $r = 0$ in polar coordinates.