Hints in the next page!

12.1. Closed answer questions.

1. If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^2(\mathbb{R}^d)$ is it necessarily true that $f \in L^2(\mathbb{R}^d)$?
2. Is the function $\frac{1}{1+ix^4}$ in the Schwartz class $\mathcal{S}(\mathbb{R})$?
3. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue[1] of $\mathcal{F}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, then necessarily $\lambda \in \{\pm 1, \pm i\}$.
4. Let A be an invertible $d \times d$ matrix with real entries. Compute the Fourier transform of $x \mapsto f(Ax)$ in terms of $\hat{f}$ and A.
5. Using the computations of the previous problem sets, compute the Fourier transform of $1/(1+4x^2)$ (which is in L^1) and of $1/(1+ix)$ (which is in L^2).

12.2. A differential equation. Given $\phi \in \mathcal{S}(\mathbb{R})$ we consider the differential equation

$$u'(x) + u(x) = \phi(x) \text{ for all } x \in \mathbb{R}.$$

1. Show that there is a unique solution within the class of Schwartz functions.
2. Taking the Fourier transform of both sides of the equation, and then the anti-Fourier transform show that
$$u(x) := \int_{\mathbb{R}} a(\xi)\hat{\phi}(\xi)e^{i\xi x}\, d\xi,$$
is indeed a solution of the above problem, for an appropriate function $a(\xi)$ to be determined.
3. ($\star$) Check that indeed this u belongs to $\mathcal{S}(\mathbb{R})$.
4. Solve again the above ODE, this time with classical methods (multiply by e^t etc..).
5. Check that the two results you found are indeed the same.

12.3. Decay of the Fourier transform and derivatives. Let $f \in L^2(\mathbb{R}^d)$ such that it's Fourier transform decays at infinity as a negative power, i.e., for some $\alpha \geq 0$ and large $M \geq 1$ it holds

$$|\hat{f}(\xi)| \leq M|\xi|^{-\alpha} \text{ for all } |\xi| \geq 1.$$

The goal of this problem is to show that in fact (up to a modification on a zero measure set) $f \in C^k(\mathbb{R}^d)$ for all integers $k < \alpha/2d$.

1. Consider for each $R > 1$ the functions
$$f_R(x) := (2\pi)^{-d/2} \int_{B_R} \hat{f}(\xi)e^{i\xi x}\, d\xi,$$
compute $\hat{f}_R$ and show that $f_R \to f$ in $L^2(\mathbb{R}^d)$.

[1] That is to say: there exists some nonzero function $v \in L^2(\mathbb{R}^d)$ such that $\mathcal{F}v = \lambda v$.

2. Show that each $f_R \in C^\infty(\mathbb{R}^d)$ but in general $f_R \notin \mathcal{S}(\mathbb{R}^d)$.
3. Using the decay assumption on $\hat{f}$, show that $\{f_R\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^d)$, provided $\alpha > d$. Conclude that, up to re-definition on a zero measure set, in this case $f \in C(\mathbb{R})$.
4. Applying the same argument to $\partial_{x_j} f_R$, show inductively that $f \in C^k$ whenever $\alpha > d + k$.

Hints:

12.1.3 Recall that $\mathcal{F} \circ \delta_{-1} = \delta_{-1} \circ \mathcal{F} = \mathcal{F}^{-1}$, where $(\delta_\alpha f)(x) = f(\alpha x)$. Furthermore, if a linear endomorphism ϕ satisfies $p(\phi) = 0$ then all its eigenvalues μ must satisfy $p(\mu) = 0$.

12.1.5 Use the inversion theorem.

12.2.1 The equation is linear, so you have to prove that any Schwartz function w satisfying $w' + w = 0$ must be identically zero.

12.2.3 Recall that $f \in \mathcal{S}$ if and only if $\hat{f} \in \mathcal{S}$.

12.3.2 Recall that $f \in \mathcal{S}$ if and only if $\hat{f} \in \mathcal{S}$, and Schwarz function must be smooth.