Hints in the next page!

13.1. Closed answer questions.

1. If $u \in L^1(\mathbb{R})$, is it necessarily true that $f(x_1, x_2) = u(x_1 + x_2)u(x_1 - x_2) \in L^1(\mathbb{R}^2)$?
2. Let (X, d) be a metric space and let $q \in X$. Assume you have a sequence $\{x_k\} \subset X$ such that *any* subsequence $\{x_{k_j}\}$ must possess a sub-subsequence $\{x_{j_{k_\ell}}\}$ such that $x_{j_{k_\ell}} \to q$ as $\ell \to \infty$. Is it true that $x_k \to q$ as $k \to \infty$?
3. Is it true that if A is a linear endomorphism of $\mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d)$, then $f(Ax) \in L^1(B_1)$?
4. Let $u \in \mathcal{S}(\mathbb{R})$, show that $\|\hat{u}\|^4_{L^4(\mathbb{R})} = 2\pi \|u * u\|^2_{L^2(\mathbb{R})}$.
5. Consider the set
$$X := \{u \in L^2(\mathbb{R}^d) : \hat{u}(\xi) \equiv 0 \text{ for almost every } |\xi| > 2\},$$
show that X is closed and not open in $L^2(\mathbb{R}^d)$.
6. Consider the set
$$X := \{u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : \int_{\mathbb{R}} u = 0\},$$
show that X is dense in $L^2(\mathbb{R})$. What happens if we consider the exact same question in $[-\pi, \pi]$ instead of $\mathbb{R}$?

13.2. Differential operators with constant coefficients. Let $u \in \mathcal{S}(\mathbb{R}^d)$ be a scalar function and $V \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)$ be vector field[1]. Prove the following formulas

$$\mathcal{F}(\text{div } V)(\xi) = i\xi \cdot \hat{V}(\xi), \quad \mathcal{F}(\text{rot } V)(\xi) = i\xi \times \hat{V}(\xi) \text{ (here assume } d = 3),$$
$$\mathcal{F}(\Delta u)(\xi) = -|\xi|^2 \hat{u}(\xi) \quad \mathcal{F}(\nabla u)(\xi) = i\xi \hat{u}(\xi), \quad \mathcal{F}(\nabla V) = i\hat{V}(\xi) \otimes \xi,$$

here ∇V is the $d \times d$ matrix $\{\partial v_i / \partial x_j\}_{i,j}$ and for two (column) vectors a, b we set $a \otimes b = ab^T = \{a_i b_j\}_{i,j}$. In $\mathbb{R}^3$ we are also using the classical cross product.

Give now quick proofs of the following claims

1. if $V \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ such that $|\text{rot } V| = \text{div } V = 0$, then $V \equiv 0$.
2. If $u \in \mathcal{S}(\mathbb{R}^d)$ is harmonic ($\Delta u = 0$), then it is identically zero.
3. If $u \in \mathcal{S}(\mathbb{R}^d)$ then
$$\int_{\mathbb{R}^d} \sum_{j,k} |\partial_{jk} u(x)|^2 \, dx = \int_{\mathbb{R}^d} (\Delta u(x))^2 \, dx.$$

[1] The Fourier of a vector field is taken component-wise, i.e., $\hat{V}(\xi) = (\hat{v}_1(\xi), \dots, \hat{v}_d(\xi))$.

13.3. Poisson Kernel in the half-space. We consider the problem of extending a function $f \in \mathcal{S}(\mathbb{R}^d)$ to an harmonic function $u\colon (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ such that $u(0+,x) = f(x)$ and $u(+\infty, x) = 0$, that is

$$\begin{cases} \Delta u = 0 & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0+,x) = f(x) & \text{for all } x\in\mathbb{R}^d, \text{ where } f\in\mathcal{S}(\mathbb{R}^d) \text{ is given }, \\ u(+\infty,x) = 0 & \text{for all } x\in\mathbb{R}^d, \end{cases}$$

1. Find the solution formally, taking the Fourier transform of u with respect to the space variable x and solving the resulting ODE. Express your solution as

   $$u(t,x) := \int_{\mathbb{R}^d} K(t,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi, \qquad \text{for all } t>0, x\in\mathbb{R}^d.$$

   for a function $K(\cdot,\cdot)$ to be specified.

2. Show that the function u defined in the previous point attains the boundary value in the following sense

   $$\lim_{t\downarrow 0}\|u(t,\cdot)-f\|_{L^\infty(\mathbb{R}^d)} = \lim_{t\uparrow\infty}\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = 0.$$

   Show that you don't need $f\in\mathcal{S}(\mathbb{R}^d)$, but only $\hat{f}\in L^1(\mathbb{R})$.

3. Show that $u\in C^\infty((0,\infty)\times\mathbb{R}^d)$, that for all $m\in\mathbb{N}, \alpha\in\mathbb{N}^d$

   $$\partial_t^m\partial_x^\alpha u(t,x) = \int_{\mathbb{R}^d} \partial_t^m K(t,\xi)(i\xi_1)^{\alpha_1}\dots(i\xi_n)^{\alpha_n}e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi,$$

   and finally that $\Delta u = 0$. Also here only $\hat{f}\in L^1$ is needed.

4. In the case $d=2$ show that, more explicitly, it holds

   $$u(t,x) = (P_t * f)(x), \text{ where } P_t(x) := \frac{1}{\pi}\frac{t}{x^2+t^2}, \text{ for all } t>0, x\in\mathbb{R}.$$

   Show that for all $p\in[1,\infty]$ and $t>0$ it holds $\|u(t,\cdot)\|_{L^p(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})}$.

Hints

13.1.1 Try to change variable and use Fubini...

13.1.2 Argue by contradiction to extract an impossible subsequence...

13.1.3 In general $f \circ A$ is measurable but depends on the particular representative we chose for f...

13.1.4 Use Plancherel and the fact that the Fourier transform sends product into convolutions...

13.1.5 Use the fact that the Fourier transform in L^2 is an isometry, hence an homeomorphism...

13.1.6 Show that $\mathcal{F}(X)$ is dense in $L^2(\mathbb{R}_\xi)$... In the case of the interval use the Fourier series and Parseval's identity...

13.2.1 the two conditions tells that $\xi \times \hat{V}(\xi) = 0$ and $\xi \cdot \hat{V}(\xi) = 0$... then we recall the geometric meaning of the cross and the dot product...

13.2.3 Use Plancherel's formula and the fact that the Fourier transform sends derivatives in multiplications by coordinate functions...

13.3.1 $K(t, \xi) = e^{-|\xi| t}$...

13.3.2 Use the explicit form of K and the dominated convergence theorem.

13.3.3 Use the explicit form of K and the differentiation theorem under the integral sign.

13.3.4 In this case we can explicitly compute the inverse Fourier transform of $e^{-|\xi| t}$ for fixed $t > 0$... For the second part recall Young's inequality and the fact that $\|P_t\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} P_t = \hat{P}_t(0) = 1$...

13. Solutions

Solution of 13.1:

Solution of 13.2:

Solution of 13.3: