2.1. Modes of convergence of measurable functions.

1. Is it true that $\lim_{\epsilon\downarrow 0}\sum_{j\geq 1}\frac{e^{\sin(\epsilon j)}-1}{j^2\arctan(\epsilon j)}=\sum_{j\geq 1} j^{-2}$?
2. Let f_k be a sequence of measurable functions such that $f_k\to f$ almost everywhere and $\sup_k\|f_k\|_{L^2(0,1)}\leq M<\infty$. Then $f_k\to f$ in $L^2(0,1)$? And in $L^1(0,1)$?
3. You know that $f_k\to f$ in $L^2(0,1)$ as $k\to\infty$. Is it true that for almost every $x\in(0,1)$ we have $f_k(x)\to f(x)$ as $k\to\infty$?

2.2. Scalar products and Hilbert spaces. Prove or disprove whether the following pairs (vector space, bilinear form) are Hilbert spaces. Additionally, write down what the squared norm of a vector is in each case.

1. $V:=\{n\times n$ matrices with complex entries$\}$ and $\langle X,Y\rangle:=\mathrm{Tr}(XY^\dagger)$.
2. $V:=L^2(\mathbb{R};\mathbb{C})$ and $\langle u,v\rangle:=\int_{\mathbb{R}}u(t)\bar{v}(t)\frac{dt}{1+t^2}$
3. $V:=\{$real polynomials of degree at most $N\}$ and[1] $\langle p,q\rangle:=p(\frac{d}{dx})|_{x=0}q$
4. $V:=L^1(0,1)$ and $\langle u,v\rangle:=\int_0^1 u(x)v(x)\,dx$.
5. $V:=H_1\times H_2$, where $(H_1,\langle\cdot,\cdot\rangle_1),(H_2,\langle\cdot,\cdot\rangle_2)$ are two Hilbert spaces, and $\langle\cdot,\cdot\rangle:=\langle\cdot,\cdot\rangle_1+\langle\cdot,\cdot\rangle_2$.
6. $V:=\mathbb{Q}^d$ and $\langle x,y\rangle:=\sum_{k=1}^d x_k y_k$.

2.3. Legendre Polynomials I. Consider the Hilbert space $H:=L^2((-1,1),dx)$. Apply the Gram-Schmidt algorithm to the ordered set $\{1,x,x^2\}\subset H$, and find three orthonormal polynomials $e_0(x),e_1(x),e_2(x)$.

2.4. Legendre Polynomials II. Consider in the Hilbert space $H:=L^2((-1,1),dx)$ the polynomials

$$P_0:=1,\quad P_k(x):=D^k((x^2-1)^k)\text{ for }k\geq 1,$$

where $D:=d/dx$. Our first goal is to prove that $\{P_j\}_{j\geq 0}$ is an orthonormal system. You can follow this outline

1. Show that each P_k has degree k and compute explicitly the polynomial $D^kP_k(x)$.
2. Show that for $0\leq k'<k$ the function $D^{k'}((x^2-1)^k)$ vanishes at ± 1 (Hint: use the Leibniz formula $D^k(f\cdot g)=\sum_{j=0}^k\binom{k}{j}D^jf\cdot D^{k-j}g)$;
3. Use the previous point and multiple integration by parts to show that if $0\leq k<k'$ then

$$\int_{-1}^1 P_k(x)P_{k'}(x)\,dx=0.$$

[1] If $p(X)$ is a polynomial, then $p(\frac{d}{dx})|_{x=0}$ is the differential operator obtained replacing X by $\frac{d}{dx}$ and then evaluating at $x=0$. Example: if $p(X)=X^2+3$ then $p(\frac{d}{dx})|_{x=0}q=q''(0)+3q(0)$.

In order to have a orthonormal basis we need to compute $\|P_k\|_{L^2(-1,1)}$. You can follow this outline

1. Given for granted[2] that $B(k+1,k+1) := \int_0^1 s^k(1-s)^k\,ds = \frac{(k!)^2}{(2k+1)!}$, show that

$$\int_{-1}^{1}(x^2-1)^k\,dx = (-1)^k\frac{2^{2k+1}(k!)^2}{(2k+1)!}$$

2. Using multiple times integration by parts and the previous points show that

$$\|P_k\|^2_{L^2(-1,1)} = \int_{-1}^{1} P_k(x)^2\,dx = (-1)^k\int_{-1}^{1}(x^2-1)^k D^k P_k(x)\,dx = \frac{2^{2k+1}(k!)^2}{2k+1}$$

3. Prove that $B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$ for all $n, m \geq 1$. You might want to prove it first for $B(0,m)$ and then find a formula (integrating by parts) that relates $B(n,m)$ with $B(n-1,m+1)$ and proceed inductively.

Finally, (double) check that indeed

$$e_0(x) = \frac{P_0(x)}{\|P_0\|_{L^2(-1,1)}}, \quad e_1(x) = \frac{P_1(x)}{\|P_1\|_{L^2(-1,1)}} \text{ and } e_2(x) = \frac{P_2(x)}{\|P_2\|_{L^2(-1,1)}},$$

where e_0, e_1, e_2 are the polynomials of exercise 2.3. Is that a coincidence that they are the same?

[2]This is a value of the so-called Euler's Beta function $B(x,y) := \int_{-1}^{1} t^{x-1}(1-t)^{y-1}\,dt = B(y,x)$.