3.1. Closed Answer questions.

1. Are $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ isometrically isomorphic as Hilbert spaces?
2. Is $C^1 \cap L^2(\mathbb{R}^d)$ dense in $L^2(\mathbb{R}^d)$?
3. If P_{K_1}, P_{K_2} are the metric projections onto two convex closed sets K_1, K_2 in some Hilbert space H, is it true that $P_{K_1 \cap K_2} = P_{K_1} \circ P_{K_2}$?
4. Given $u \in L^2(0,1)$ there exist a unique polynomial $\bar{p}$ such that $p \mapsto \|u - p\|_{L^2(0,1)}$ is minimal. True or false?

3.2. Projection on subspaces. For each of the following pairs (H, V) where H is an Hilbert space and V is a subspace discuss whether V is closed or not and give a formula for the orthogonal projection $\pi\colon H \to \overline{V}$. Footnotes contain hints.

1. $H = L^2(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) < \infty$, and $V = \{u \in H : u \equiv \text{ const. } \mu\text{-a.e.}\}$[1].
2. $H = L^2(-1,1)$ and $V = \{u \in H : u \text{ is } \mathcal{M}_{sym}\text{-measurable}\}$[2] where the σ-algebra $\mathcal{M}_{sym}$ is the σ-algebra generated by
$$\{E \text{ Lebesgue measurable and } E = -E\}.$$
3. $H = L^2(0,1)$ and $V = \mathbb{R} \log x$.[3]
4. $H = L^2(\mathbb{R}^3; \mathbb{R}^3)$ and $V = \{\vec{u} \in H : \vec{x} \cdot \vec{u}(x) = 0 \text{ for a.e. } x \in \mathbb{R}^3\}$.[4]
5. $H = L^2(\mathbb{R}^d)$ and V is the subspace of radial functions i.e.,
$$V = \{u \in H : \exists U \in L^1_{loc}(0,\infty) \text{ such that } u(x) = U(|x|) \text{ for a.e. } x \in \mathbb{R}^d\}$$[5]

3.3. Projection on convex sets. For each of the following pairs (H, K) where H is an Hilbert space and K is a convex set (check it, if it is not clear) discuss whether K is closed or not and give a formula for the metric projection $\pi\colon H \to \overline{K}$.

1. $H = L^2(0,1)$ and $K = \{u \in H : u > 0 \text{ a.e.}\}$.
2. $H = L^2(0, 2\pi)$ and $K = \{u \in H : u \geq \sin(\cdot) \text{ a.e.}\}$.
3. $H = \mathbb{R}^2$ and $K = [-1,1] \times [-1,1]$.
4. $H = L^2(0,1)$ and $K = \{u \in H : \int_0^1 u\phi \leq 0\}$, where $\phi \in L^2$ is given.

3.4. The tight fishball. Let H be a real Hilbert space and $U \subset H$ be a bounded, nonempty set. Show that, among all the closed balls which contain U, there is only one with minimal radius.[6]

[1] You can use the definition, and minimize the function $\mathbb{R} \ni t \mapsto ⨍_\Omega |u(x) - t|^2 \, d\mu$
[2] There is a simpler characterization of V...
[3] Here, as in item 1, V has dimension 1.
[4] Everything is happening in the co-domain, so you can use the definition...
[5] Work in polar coordinates and use item 1. on each spherical shell.
[6] Look at a sequence of minimizing balls $B_{r_k}(x_k)$ and try to prove that $\{x_k\}$ is Cauchy with the parallelogram identity. It helps to do a picture in 2D, i.e. work out the case $H = \mathbb{R}^2$ first.