4.1. Closed answer questions.

1. In order to prove that a linear map $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is continuous it is enough to prove that $\|Tu\|_{L^2(\mathbb{R})} \leq 100$, provided $u \in L^2(\mathbb{R})$ and $\|u\|_{L^2(\mathbb{R})} \leq 7$. True or false?
2. Give an example of a nonzero continuous linear functional on $L^2(0,1)$.
3. Alice is given a bounded linear functional $\phi \in L^2(0,1)^*$, and Bob is given a bounded linear functional $\psi \in L^2(0,1)^*$. They check that $\phi(u) = \psi(u)$ for all $u \in C([0,1])$. Is it necessarily true that $\phi = \psi$?
4. If ϕ is a continuous linear functional on an Hilbert space H, then $\ker\phi$ is a closed vector subspace of H. True or false?
5. The inequality "$\|(u_k)\|_{\ell^2(\mathbb{N})} \leq \|(u_k)\|_{\ell^1(\mathbb{N})}$ for all sequences (u_k)", can be equivalently phrased as "the inclusion $\ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$ is 1-Lipschitz". True or false?

4.2. Norm of the multiplication operator. For $u \in H := L^2(0,1)$ consider the operator

$$M_a\colon u(x) \mapsto a(x)u(x),$$

where $a\colon (0,1) \to \mathbb{R}$ is a given measurable function. We want to prove that M_a is continuous from H in itself if and only if $a \in L^\infty(0,1)$, in which case $\|M_a\|_{\mathcal{L}(H)} = \|a\|_{L^\infty(0,1)}$.

1. Given for granted the claim, $\|M_{\exp}\|_{\mathcal{L}(H)} = \ldots$?
2. Prove the inequality
$$\int_0^1 a^2(x)u^2(x)\,dx \leq \sup_{(0,1)} |a|^2 \int_0^1 u^2(x)\,dx,$$
and deduce that $\|M_a\|_{\mathcal{L}(H)} \leq \|a\|_{L^\infty(0,1)}$.
3. Show that if $E \subset (0,1)$ is measurable with $|E| > 0$, then
$$\frac{\|M_a \mathbf{1}_E\|^2_{L^2(0,1)}}{\|\mathbf{1}_E\|^2_{L^2(0,1)}} = \frac{1}{|E|}\int_E a^2(x)\,dx.$$
4. Choosing properly the measurable set E in the previous point[1], prove that $\|M_a\|_{\mathcal{L}(H)} \geq \|a\|_{L^\infty}$.

[1]Hint: try with $E =$"the set where $|a|$ is large"...

4.3. Bounded Linear Operators I. Prove that each of the following linear operators is bounded from $\ell^2(\mathbb{N})$ in itself [2]. Draw the infinite matrix that represents each of them.

1. (Shift operator) $S\colon (u_0, u_1, u_2, \dots) \mapsto (0, u_0, u_1, \dots)$.
2. (Diagonal matrix) $M_\lambda\colon (u_0, u_1, u_2, \dots) \mapsto (\lambda_0 u_0, \lambda_1 u_1, \lambda_2 u_2, \dots)$, where $\{\lambda_j\}_{j\geq 0}$ is some given sequence such that $\sup_{j\geq 0}|\lambda_j| = 7$.
3. $T\colon (u_0, u_1, u_2, \dots) \mapsto (u_0 - u_1, u_1 - u_2, u_2 - u_3, \dots)$.
4. (Hilbert-Schmidt matrix) For each $k \geq 0$ set $(Au)_k := \sum_{j\geq 0} A_{k,j} u_j$, where the infinite matrix $\{A_{i,j}\}_{i\geq 0, j\geq 0}$ satisfies [3]
$$\sum_{i,j} |A_{i,j}|^2 < +\infty.$$

4.4. Bounded linear operators II. Prove the following inequalities and interpret them as the continuity of a suitable linear map between suitable normed vector spaces:

1. For all $u \in L^2(\mathbb{R})$ it holds
$$\int_0^1 u^2(t)\,dt \leq \int_{\mathbb{R}} u^2(t)\,dt.$$
2. For each polynomial $p(X) = p_0 + p_1 X + \ldots + P_k X^N$ it holds
$$\max_{x\in[-1,1]} |p(x)| \leq \sum_{j=0}^{N} |p_j|.$$
3. For all $u \in C^1([0,1])$ with $u(0) = 0$ it holds[4]
$$\max_{x\in[0,1]} |u(x)| \leq \int_0^1 |u'(t)|\,dt.$$

[2]Concretely, you have to establish that the ℓ^2 size of the image of any sequence (u_k) is bounded by a multiple of the ℓ^2 size of (u_k) itself.

[3]Hint: for each k: $\big(\sum_{j\geq 0} A_{k,j} u_j\big)^2 \leq \big(\sum_{j\geq 0} A_{k,j}^2\big)\big(\sum_{j\geq 0} u_j^2\big)$, by Cauchy Schwarz...

[4]Hint: use the fundamental Theorem of Calculus, i.e., that a function is the integral of its derivative...