Recall that if $f \in L^1((-\pi,\pi),\mathbb{C})$ and $k \in \mathbb{Z}$ then the k^{th} Fourier coefficient is the complex number defined by

$$c_k(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

Hints for the exercises in the next page!

5.1. Closed answer questions.

1. Is the unit ball of $L^\infty(0,1)$ closed in the $L^2(0,1)$ topology?
2. Let $V \subset H$ be a closed subspace, $u \in H$, $\tilde{u} \in V$ and assume

$$\langle u-\tilde{u}, v-\tilde{u}\rangle = 0 \text{ for all } v \in D, \tag{1}$$

   where $D \subset V$ is dense. Is it true that $\tilde{u} = P_V(u)$?
3. Is the inclusion map

$$\iota\colon (L^\infty(0,1), \|\cdot\|_{L^1(0,1)}) \to (L^\infty(0,1), \|\cdot\|_{L^2(0,1)}), \quad u \mapsto u,$$

   bounded?
4. Let $p(X,Y)$ be a polynomial in two variables and let $f(x) := p(\cos(x),\sin(x))$. If it true that $c_k(f) \neq 0$ only for finitely many values of k? (I.e., is f a trigonometric polynomial?)
5. Compute the Fourier series of $e^{-|x|}$ and $\sin(x/3)$ (they are not particularly nice, but try *to get the computation right*!).

5.2. Legendre polynomials III.

1. Using the Stone-Weierstrass Theorem, prove that polynomial functions are dense in $L^2(-1,1)$.
2. Recall the Legendre polynomials $P_k(x) := D^k((x^2-1)^k)$. Show that $\operatorname{span}\{P_k\}_{k\in\mathbb{N}} = \operatorname{span}\{x^k\}_{k\in\mathbb{N}}$, so - by the previous point - Legendre polynomials have dense span. Hence, combining this with exercise 2.4, they form a complete orthogonal system.

5.3. Fourier series of x^m.

1. Show that $c_k(1) = \sin(\pi k)/(\pi k)$ for all $k \in \mathbb{Z}\setminus\{0\}$. Notice that the identity holds also for all $k \in \mathbb{R}$ (including $k=0$!).
2. Consider $k \mapsto c_k(f)$ as a function of $k \in \mathbb{R}$, for a fixed function $f \in L^1(-\pi,\pi)$. Show the identity $c_k(xf) = i\frac{d}{dk}c_k(f)$.
3. Compute for each $m \in \mathbb{N}$ the Fourier series of x^m.

Hints:

5.1.1 The answer is positive. Use the sequential characterization of closeness, and the fact that if a sequence converges in L^2, then an appropriate subsequence converges a.e. (this is Theorem 2.4.4 in Da Lio's notes).

5.1.2 The answer is positive. Thanks to the theory, what is missing in order to characterize $\tilde{u}$ as $P_V(u)$ is that (1) should hold for all $v \in H$ rather than all $v \in D$, but D is dense...

5.1.3 The answer is negative. Try to build functions for which the L^1 norm stays bounded, while the L^2 norm explodes...

5.1.4 The answer is positive. Recall the identity $2\cos(x) = e^{ix} + e^{-ix}$, and use it to express $\cos(x)^m$...

5.2.1 Mimic the proof of the completeness of the Fourier basis, as was done in class. In this case it is actually simpler!

5.2.2 One inclusion is easy, for the other one show inductively on N that

$$x^N \in \operatorname{span}\{P_0, P_1, \ldots, P_N\}.$$

In order to do so, notice that $P_k(x) = \frac{(2k)!}{k!}x^k + \{\text{lower order terms}\}$, so the leading-order coefficient on P_N is nonzero...

5.3.2 Write the definition of $c_k(f)$ and commute the derivative in k with the integral in x.

5.3.3 Use 5.3.1, 5.3.2 and the analytic expansion $\sin(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell+1}}{(2\ell+1)!}$. Recall that the N^{th} derivative of a function is $N!$ times the N^{th} coefficient of its analytic expansion.