**Hints in the next page!**

**6.1. Closed answer / quick questions.**

1. Let $(H, \langle \cdot, \cdot \rangle)$ be an Hilbert space and $V \subset H$ a proper dense subspace. Can $(V, \langle \cdot, \cdot \rangle)$ be an Hilbert space, at least in some examples?
2. Is the space of sequences with only finitely many nonzero terms, dense in $\ell^2(\mathbb{N})$?
3. Find the Fourier coefficients of $\sin^3(x)$ (compute no integrals!).
4. If $f_n \to f$ and $g_n \to g$ in $L^2(0,1)$, then necessarily $f_n g_n \to fg$ in $L^1(0,1)$?
5. If $f_n \to f$ in $L^1([-\pi,\pi];\mathbb{C})$ then $c_k(f_n) \to c_k(f)$ as $n \to \infty$, uniformly in $k$?

**6.2. Fourier series in $(0, \pi)$.** We want to show that every function in $L^2([0,\pi];\mathbb{R})$ can be expressed as a real Fourier series of sines.

1. Show that if $f \in L^2([-\pi,\pi];\mathbb{C})$ is odd then $c_k(f)$ are purely imaginary and $c_0 = 0$;
2. Show that if $f \in L^2([-\pi,\pi];\mathbb{R})$ is odd then its Fourier series simplifies to
$$S_N f(x) = \sum_{1 \le k \le N} \underbrace{2ic_k(f)}_{\in \mathbb{R}} \sin(kx)$$
3. Given $g \in L^2([0,\pi];\mathbb{R})$ show that $\tilde{S}_N g \to g$ in $L^2$ where
$$\tilde{S}_N g(x) := \sum_{1 \le k \le N} \tilde{a}_k(g) \sin(kx), \qquad \tilde{a}_k(g) := \frac{2}{\pi} \int_0^\pi g(x) \sin(kx)\, dx \in \mathbb{R}.$$
4. Conclude that $\{\sqrt{2/\pi} \sin(kx)\}_{k \ge 1}$ in an Hilbert basis for $L^2([0,\pi];\mathbb{R})$.

**6.3. Uniqueness of coefficients in $L^1$.** Fix $f \in L^1([-\pi,\pi];\mathbb{C})$, and let $c_k = c_k(f)$ be its Fourier coefficients, we want to show that if $c_k(f) = 0$ for all $k \in \mathbb{Z}$, then $f \equiv 0$ a.e..

1. Show that if actually $f \in L^2([-\pi,\pi];\mathbb{C})$, then the statement follows directly from a Theorem seen in class.
2. Show that if $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in L^\infty((-\pi,\pi);\mathbb{C})$, then we must have $f = 0$ a.e..
3. Show that if $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in C_c((-\pi,\pi);\mathbb{C})$, then we must have $f = 0$ a.e..
4. Using and appropriate density result seen in class, show that if $c_k(f) = 0$ for all $k$, then indeed $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in C_c((-\pi,\pi);\mathbb{C})$. Hence by the previous steps $f = 0$.

**6.4. Coefficients summability implies convergence.** Let $f \in L^1([-\pi,\pi];\mathbb{C})$, and let $c_k = c_k(f)$ be its Fourier coefficients.

1. Show that if $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$, then in fact $f \in L^2([-\pi,\pi];\mathbb{C})$;

2. Show that if $\sum_{k\in\mathbb{Z}} |c_k| < \infty$, then in fact $f \in C_{per}([-\pi,\pi];\mathbb{C})$[1]

**Hints:**

6.1.1. In order to be an Hilbert space $V$ should be complete, and it seems odd that a proper dense space contains the limits of all its Cauchy sequences...

6.1.2. Yes it is, to prove this you have to approximate any given element of $\ell^2$ with a sequence with finitely many nonzero elements... try the first that comes to you mind...

6.1.3. Recall that $2i\sin(x) = e^{ix} - e^{-ix}$, so if you take cubes of both sides...

6.1.4. It is true, start with the bound

$$|f_n g_n - fg| \leq |f_n - f||g_n| + |f||g - g_n|,$$

and use Cauchy-Schwarz.

6.1.5. This is true, try estimate to $|c_k(f) - c_k(f_n)|$ with $\|f - f_n\|_{L^1}$, independently from $k$...

6.3.1. Recall Parseval's identity

6.3.2. Try what happens setting $\phi := \bar{f}/(1+|f|^2)$...

6.3.3. We would like to set again $\phi = \bar{f}/(1+|f|^2)$, but $f$ is not continuous... nevertheless $C_c(-\pi,\pi)$ is dense in $L^1(-\pi,\pi)$...

6.4.4. If you take any set of complex coefficients $a_{-N},\dots,a_N$, we obviously have

$$a_{-N}c_N(f) + \ldots + a_N c_N(f) = 0,$$

which can be rewritten as $\int f\phi = 0$ for a $\phi$ which is of a very particular kind...

6.4.1. First show that the partial sums $S_N(x) := \sum_{-N}^{N} c_k(f)e^{ikx}$, are a Cauchy sequence in $L^2$ (use Parseval's identity). Hence deduce that $S_N \to \tilde{f}$ in $L^2$, for some $\tilde{f}$ in $L^2$. Now use Problem 6.3 to show that we must in fact have $\tilde{f} = f$ as $L^1$ functions.

6.4.2. First show that the partial sums $S_N(x) := \sum_{-N}^{N} c_k(f)e^{ikx}$, are a Cauchy sequence in $C_{per}([-\pi,\pi])$ (the series is absolutely convergent, thanks to the summability assumptions on the $\{c_k(f)\}$). Hence deduce that $S_N \to \tilde{f}$ uniformly, for some $\tilde{f}$ in $C_{per}$. Now use Problem 6.3 to show that we must in fact have $\tilde{f} = f$ as $L^1$ functions.

[1]This is a slight abuse of terminology. More precisely: there exist a (necessarily unique) continuous and periodic $\tilde{f}$ such that $\tilde{f} = f$ a.e.