8.1. Closed answer questions.

1. Construct $f\colon [-\pi,\pi] \to \mathbb{R}$ which is continuous, but not Hölder at $\bar{x}=0$.
2. Let V be the vector space of sequences $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that
$$\|f\|_V := \Big\{\sum_{k\geq 1} k^2|f(k)|^2\Big\}^{1/2} < \infty.$$
Can you choose a scalar product on V that makes V an Hilbert space?
3. Let V be the vector space of sequences $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that
$$\|f\|_V := \sum_{k\geq 1} k|f(k)| < \infty.$$
Can you choose a scalar product on V that makes V an Hilbert space?
4. Explain the difference between the following spaces of (real) functions and provide elements that fit in one but none of the others:
$$C_{per}([-\pi,\pi]), \quad C^2_{per}([-\pi,\pi]), \quad C((-\pi,\pi)), \quad C([-\pi,\pi]).$$

8.2. Fourier series convergence recap. For each of the following functions f on $[-\pi,\pi]$,
$$\tan(\sin(x)); \quad |x|^{-1/2}; \quad |x|^{2/3}; \quad x; \quad e^{-x^2}; \quad (x^2-\pi^2)^2;$$
answer to the following questions using the Theorems seen in class (Achtung! If you cannot apply any of those Theorems, that's still a valid answer)

1. Are the Fourier coefficients well defined?
2. Is it true that $S_N(f) \to f$ in L^2?
3. Is it true that $S_N(f)(x) \to f(x)$ for all $x \in (-\pi,\pi)$? What about $x = \pm\pi$?
4. Is it true that $S_N(f) \to f$ in C_{per}?
5. If possible, give two non-negative values of $0 \leq \alpha_1 < \alpha_2$ such that
$$\sum_{k\in\mathbb{Z}} |k|^{\alpha_1}|c_k(f)| < +\infty, \text{ but } \sum_{k\in\mathbb{Z}} |k|^{\alpha_2}|c_k(f)| = +\infty.$$

8.3. The Dirichlet kernel is not in L^1. Recall that $D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$, for all $n \geq 1$ and $x \in \mathbb{R}$, is a 2π periodic function.

1. Using $|\sin(t)| \leq |t|$, then changing variables and then dividing the domain of integration, show that

$$\int_0^\pi |D_n(x)|\,dx > 2\sum_{j=0}^{n-1}\int_{j\pi}^{(j+1)\pi} |\sin(y)|\frac{dy}{y}.$$

2. Show that for each $j \geq 0$ it holds

$$\int_{j\pi}^{(j+1)\pi} |\sin(y)|\frac{dy}{y} \geq \frac{c}{j+1},$$

   for some (explicit) constant $c > 0$.

3. Conclude that $\|D_n\|_{L^1(0,\pi)} \geq O(\log n)$ as $n \to \infty$.

8.4. Fourier series of the product. Let $f, g \in L^2([-\pi,\pi];\mathbb{C})$, prove that

$$c_k(fg) = \sum_{j\in\mathbb{Z}} c_j(f)c_{k-j}(g) \qquad \text{for all } k \in \mathbb{Z},$$

and in particular that $c_k(fg)$ is well-defined, and that the series at the RHS is absolutely convergent.

Hints:

8.1.1. Try with $1/|\log t|$...

8.1.2. See the given norm as the L^2 norm on a suitable abstract measure space...

8.1.3. See the given norm as the L^1 norm on a suitable abstract measure space. Then recall that Hilbert norms must satisfy certain identities...

8.1.4. Can a continuous function oscillate madly as it approaches the boundary?

8.2.1. Given our definitions, this is like checking whether $f \in L^1$.

8.2.2. By Parseval's Theorem, this is like checking whether $f \in L^2$.

8.2.3. Convergence at x is granted whenever the function is Hölder at x... there is one case where this question cannot be answered directly with what we have seen!

8.2.4. Recall that for piecewise C^1 functions the Fourier series converges uniformly... there is one case where this question cannot be answered directly with what we have seen!

8.2.5. Notice that if one of those sums is finite than necessarily f is in C_{per}. In the remaining cases play two games: if $\alpha_1 > N$ then f has at least N *continuous and periodic* derivatives (C^N_{per})... On the other hand if f has at least N continuous derivatives, and the first $N-1$ are periodic, then $\{|k|^N c_k(f)\} \in \ell^2$, which implies $\{|k|^{N-\epsilon} c_k(f)\} \in \ell^1$.

8.3.3. Remember that the harmonic series $H_n := \sum_{k\geq 1}^{n} 1/k$ diverges and, more precisely, $H_n \asymp \log n$.

8.4 Prove the formula first in the case in which both f and g are trigonometric polynomials. Then, in the general case, argue by approximation in L^2. Try to justify the limit procedures carefully, you need no more than the dominated convergence and Hölder's inequality.