Hints in the next page!

9.1. Closed answer questions.

1. If $f \in C^2(\mathbb{R})$ and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$, then necessarily $f|_{[-\pi,\pi]} \in C^2_{per}$? What about the viceversa?
2. For which values of $\alpha \in \mathbb{R}$ and $p \geq 1$ we have that $\{k^\alpha\} \in \ell^p(\mathbb{N} \setminus \{0\})$?
3. Does it exists a continuous and periodic function f such that $c_k(f) \asymp |k|^{-1/3} \log |k|$ as $k \to \infty$?
4. Does it exist a function $f \in L^1(-\pi, \pi)$ such that $c_k(f) \not\to 0$ as $|k| \to \infty$?
5. Give an example of a C^∞ and 2π-periodic function which is not a trigonometric polynomial. Can you make it analytic?

9.2. Formal solutions of PDEs. For the following PDEs of evolution type try to find the most general solution of the form $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{-ikx}$, without worrying about convergence issues. Of course the functions $\{u_k(t)\}_{k\in\mathbb{Z}}$ might depend on the Fourier coefficients of $u(0,\cdot)$ (and sometimes also of $\partial_t u(0,\cdot)$)

1. $\partial_t u = \cos(t)\partial_{xx} u$
2. $\partial_{tt} u - \partial_{xx} u = 0$
3. $\partial_t u = \frac{1}{1+t^2} u + \partial_{xx} u$

For each of these cases write down an example of solution which is not a constant.

9.3. Free Schrödinger equation in a ring. Consider the evolution problem with periodic boundary conditions:

$$\begin{cases} i\partial_t u + \partial_{xx} u = 0 & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(t,x) = u(t, x+2\pi) & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } f \in C^\infty(\mathbb{R}), 2\pi\text{-periodic.} \end{cases}$$

1. Explain why solutions cannot be purely real-valued, unless they are constant.
2. Explain why, for each fixed large N, we have $\sup_k |k|^N |c_k(f)| < \infty$.
3. Write the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, where the $\{u_k(t)\}$ depend on the Fourier coefficients of f.
4. Show that the formal solution is in fact a true solution and is C^∞ in both variables.
5. Show that we found the only possible solution: if v is a solution of the problem which is C^2_{per} in space and C^1 in time, then $u = v$.
6. Write explicitly u in the case $f = 2\cos(3x)$.
7. Does this equation enjoy the "smoothing effect" of the heat equation?

Hints:

9.1.1 These are two ways of presenting the same class of functions...

9.1.2 Recall that $\sum_{k\geq 1} k^s < \infty$ if and only if $s < -1$...(Why?)

9.1.3 The given sequence decays quite slowly... it is not even in ℓ^2...

9.1.4 Riemann-Lebesgue...

9.1.5 You might find some inspiration in exercise 8.2...

9.2 Proceed as in the case of the heat equation in the lecture notes. The ODEs you find should be solvable without ugly computations.

9.3.1 Argue by contradiction and look at the coefficients of the equation...

9.3.2 f is smooth *and* periodic...

9.3.4 You need to show that the coefficients $\{c_k(\partial_t^m \partial_x^n u(t,\cdot))\}$ are summable. This follows from the decay of the $\{c_k(f)\}$. It might be convenient to show uniform (with respect to N) bounds on the mixed derivatives of

$$u_N(t,x) := \sum_{|k|\leq N} u_k(t) e^{-ikx}.$$

9.3.5 Argue exactly as in the proof of the uniqueness for the heat equation.

9.3.6 The coefficients $u_k(t)$ have roughly the same size of the $c_k(f)$, thus we do not expect regularization...