1 Hilbert Spaces

- Inner product spaces
  - Vector space: Definition. (Definition 1.1)
  - Inner product space: Definition and properties. (Definition 1.3 & Basic properties)
  - A vector subspace of an inner product space is an inner product space itself: Statement. (Example 1.7)
  - Examples of inner product spaces: $L^2(X, \mu, \mathbb{R})$, $L^2(X, \mu, \mathbb{C})$, $\ell^2_{\mathbb{C}}$, and $\mathcal{C}([0,1])$. (Examples 1.8, 1.9 and 1.11)
  - Norm: Definition. (Definition 1.13)
  - Not every norm is induced by an inner product: Counterexample. (Example 1.19)
  - Cauchy-Schwarz inequality for complex Hilbert spaces: Statement and **proof**. (Lemma 1.15)
  - Parallelogram law: Statement. (Proposition 1.18)
  - Polarization identities: Statements. (Proposition 1.20)
  - Inner product through the polarization identities. (Remark 1.21)
  - Ptolemy's inequality: Statement. (Proposition 1.22)
  - Inner product continuity: Statement. (Remark 1.23 & Lemma 1.24)
  - Linear maps are isometries if and only if they preserve the inner product structure: Statement and **proof**. (Exercise 1.2)
- Normed vector spaces
  - Inner product spaces are normed vector spaces: Statement.
  - Examples of finite-dimensional vector spaces: $\mathbb{K}^d$ with p-norms ($1 \leq p < \infty$) and the maximum norm ($p = \infty$). (Example 1.27)
  - Examples of infinite-dimensional vector spaces: $\mathcal{C}([0,1])$ with p-norms ($1 \leq p < \infty$) and the uniform norm ($p = \infty$). (Example 1.28)
  - L^p-spaces as measure spaces are complete normed vector spaces: Statement and **proof**. (Example 1.29)
  - Open ball: Definition.
  - Convexity: Definition. (Definition 1.30)
  - Open balls are convex in a normed vector space: Statement and **proof**. (Exercise 1.3)
  - Interior points, open/closed sets, topology, topological vector space: Definitions.
  - Topological vector space: Definition. (Definition 1.31)
  - Normed vector spaces are topological vector spaces: Statement.
  - Convergence of a sequence: Definitions in the topological sense and in the metric space sense. (Definition 1.32)
  - Completeness of a metric space, Cauchy sequence: Definitions. (Definition 1.32 & Recall)

- Convergent sequences are Cauchy: Statement and **proof**. (Remark 1.33)
- Limit points: Definition. (Lemma 1.34)
- A set is closed if and only if it contains all of its limit points. Statement. (Lemma 1.34)
- Dense subsets: Definitions. (Definition 1.35)
- Equivalent norms: Definition. (Definition 1.36)
- In finite dimension, all norms are equivalent: Statement and **proof**. (Exercise 1.4)
- Equivalent norms induce the same topology: Statement. (Proposition 1.37)
- Relation between the maximum norm and the 1- and 2-norms in $\mathbb{K}^d$: Statement and **proof**. (Exercise 1.5)

- Hilbert spaces
  - Hilbert spaces can be viewed as a generalization of Euclideans spaces to infinite-dimensional settings: Statement.
  - Canonical norm and distance through an inner product: Definitions.
  - Hilbert space: Definitions. (Definition 1.39)
  - A norm is induced by a scalar product if and only if the parallelogram identity holds: Statement. (Remark 1.40)
  - Examples: $\mathbb{C}^d$, $L^2(X, \mu, \mathbb{C})$, $\ell^2_{\mathbb{C}}$ with their canonical scalar products, finite-dimensional inner product spaces. (Examples 1.41 and 1.42)
  - Subspace is a Hilbert space if and only if it is closed. Proper dense subspaces are not Hilbert spaces: Statements and **proofs**. (Exercise 1.6)
  - Inner product spaces are not necessarily complete: Counterexample. (Example 1.43)
  - Orthogonality: Definition. Relation between orthogonality and the norm in an inner product space. (Exercise 1.7)
  - Projection operator, Gram-Schmidt orthogonalization process and orthonormal basis: Definitions. (Recall: Gram-Schmidt process)

- Basis of a Hilbert space
  - Separability: Definition. (Definition 1.44)
  - Examples of separable topological spaces: $\mathbb{R}^d$, $\mathbb{C}^d$, compact metric spaces, $\mathcal{C}(K)$, L^p-spaces. (Example 1.45)
  - Orthonormal system: Definition. (Definition 1.49)
  - Bessel inequality and Parseval's identity: Statements. (Theorem 1.50)
  - Hilbert basis: Definition. (Definition 1.51)
  - Equivalence of Hilbert and algebraic bases in finite dimensions, distinction in infinite dimensions
  - Completeness criterion: Statement and **proof**. (Theorem 1.52)
  - Existence of a basis: Statement. (Theorem 1.53)

- Separable complex Hilbert spaces are isometric to $\ell^2_{\mathbb{C}}$: Statement and **proof**. (Corollary 1.56)

- Closest point property, projections
  - Projections on closed vector subspaces and closed convex sets: Statements and **proofs**. (Theorem 1.57 & Remark 1.58)
  - Orthogonal space is not trivial: Statement and **proof**. Importance of the closedness hypothesis. (Corollary 1.59 & Remark 1.60)
  - Projection over finite-dimensional and separable closed subspaces: Statements and **proofs**. (Examples 1.61 and 1.62)
  - Orthogonal complement: Definition. Closedness, non-triviality of the orthogonal complement and trivial intersection with the linear space: Statements and **proofs**. (Definition 1.63 & Remarks 1.65 and 1.66)
  - Orthogonal decomposition: Statement and **proof**. (Proposition 1.67)

- Linear operators and continuous functionals
  - Linear, bounded and unbounded operators, functionals, $L(X, Y)$ and continuous dual space: Definitions. (Definition 1.68)
  - Example of unbounded operator: the derivative operator with **proof**. (Example 1.70)
  - Operator norm: Definition.
  - Equivalence between boundedness and continuity: Statement. (Proposition 1.75)
  - Riesz Representation Theorem: Statement and **proof**. (Theorem 1.77)
  - Isomorphism between a Hilbert space and its dual: Statement. (Corollary 1.78)

2 Fourier Series

- Definitions and main properties
  - Fourier coefficient: Definition. Well-posedness in L^1 and L^2: Statement and **proof**. Fourier coefficients are bounded linear functionals on L^2: Statement. (Definition 2.1, Exercise 2.1 & Remark 2.3)
  - Fourier partial sums: Definition. (Definition 2.2)
  - Fourier Basis Theorem: Statement. (Theorem 2.4)
  - Convergence of the Fourier partial sums in L^2 and Parseval's identity: Statement. (Corollary 2.7)
  - Expressions of Fourier coefficients and partial sums using sine and cosine, and simplifications based on function parities: Statements and **proofs**. (Exercise 2.2)
  - Examples: Fourier coefficients of trigonometric functions and trigonometric polynomials. (Example 2.10 & Exercise 2.4)
  - Examples: Computation of series through Parseval's identity. (Example 2.11 & Exercise 2.5)
  - Equivalence between real-valuedness of the function and conjugation symmetry of the Fourier coefficients: Statement. (Proposition 2.12)

- Series in Banach spaces
  - Convergence criteria of series in Hilbert spaces and Pythagoras' Theorem: Statements. (Theorem 2.13)
  - Completeness and convergence in $\mathcal{C}_b^m(\Omega;\mathbb{C})$: Statement. (Example 2.15)
- Regularity and asymptotic behavior of Fourier coefficients
  - Fourier coefficients of the derivative: Statement and **proof**. Generalization to $\mathcal{C}^h$ functions: Statement. (Proposition 2.17 & Theorem 2.22)
  - Asymptotic behavior of Fourier coefficients in $\mathcal{C}^1$: Statement and **proof**. Generalization to $\mathcal{C}^h$ functions: Statement. (Proposition 2.19 & Theorem 2.22)
  - Uniform convergence of the Fourier partial sum of $\mathcal{C}^1$ functions: Statement and **proof**. Generalization to $\mathcal{C}^h$ functions: Statement. (Corollaries 2.20 and 2.24)
  - Summability of Fourier coefficients implies regularity of the function and uniform convergence of the Fourier partial sums along the derivatives: Statement. (Theorem 2.25)
- Pointwise convergence of Fourier series
  - Representation of the N^{th} Fourier partial sum as convolution with the N^{th} Dirichelt kernel: Statement. (Equation (2.19))
  - Riemann-Lebesgue Lemma: Statement. (Lemma 2.30)
  - Pointwise convergence of the Fourier partial sum at points where the function is Hölder continuous: Statement and **proof** given for granted the previous two items. (Theorem 2.27)
- Overview of convergence (tables)
  - Relation between the modes of convergence L^2, L^∞, *a.e.* and "uniform".
  - Nested classes of functions
  - Relationships between the decay properties of the Fourier coefficients and convergence of the Fourier partial sums.
- Heat equation
  - all section 2.6 in the lecture notes (with **proofs**), except the first three paragraphs "Heuristic derivation", "Transmission of the Thermal energy" and "The thermal energy is the temperature".

    In particular: strategy for solving the PDE, the existence theorem for nice initial data and the uniqueness theorem of sufficiently regular solutions. (Theorems 2.32 and 2.35)

3 Fourier Transform

- Fourier transform in $L^1(\mathbb{R}^d)$
  - Fourier transform: Definition. Well-posedness in L^1 and properties: Statements and **proofs**. (Definition 3.1 & Theorem 3.3)

- Translation, modulation, dilatation, convolution formulas: Statements and **proofs**. (Propositions 3.6 and 3.8)
- Examples: Exponential envelope function $e^{-|x|}$ and characteristic function $\mathbb{1}_{[-1,1]}$: Statements and **proofs**. Normal Gaussian distribution Φ_d: Statement. (Examples 3.9, 3.11 and 3.12 & Remark 3.10
- Fourier transform of a radial function is radial: Statement and **proof**. (Exercise 3.1)
- Parity and valuedness of the Fourier transform: Statements and **proofs**. (Exercise 3.2)

- Space of Schwartz functions
  - Fourier transform of partial derivatives: Statement and **proof**. (Proposition 3.14)
  - Derivative of the Fourier transform: Statement and **proof**. (Proposition 3.15)
  - Fourier transform of the normal Gaussian distribution through an Ordinary Differential Equation: Statement and **proof**. (Example 3.16)
  - Example: Fourier transform of $xe^{-|x|}$: Statement and **proof**. (Exercise 3.5)
  - Schwartz space: Definition. Inclusion in L^p-spaces and growth rate of Schwartz functions: Statements. (Definition 3.17 & Remark 3.18)
  - Closedness under partial derivation and multiplication by polynomials: Statement. (Remark 3.19)
  - Strict inclusion of smooth, compactly supported functions in the Schwartz space: Statement and counterexample for equality. (Remark 3.20)
  - Semi-norm and norm on Schwartz space: Statement. (Exercise 3.6)
  - Schwartz functions have smooth Fourier transforms, Fourier transform of the derivatives and derivatives of the Fourier transform: Statements. (Proposition 3.21)
  - Schwartz functions have Schwartz Fourier transforms: Statement. (Corollary 3.22)
  - Inversion formula in $\mathcal{S}(\mathbb{R}^d)$ and relaxation to functions in $L^1 \cap \mathcal{F}(L^1)$: Statements. (Theorem 3.25 & Remark 3.29)
  - Shift formula: Statement. (Lemma 3.28)
  - Injectivity of the Fourier transform: Statement. (Remark 3.30)
  - Examples: $u * u = u$ implies $u = 0$ almost everywhere, Fourier transforms of $\frac{1}{1+x^2}$, $\frac{1}{(1+x^2)^2}$ and $\frac{\sin(2x)}{1+x^2}$, and equation $u * u(x) = \frac{2}{1+x^2}$. (Exercises 3.8-3.12)

- Fourier transform in $L^2(\mathbb{R}^d)$
  - The Fourier transform is an isometry on L^2: Statement. (Theorem 3.31)
  - Fourier transform in L^2: Definition. Well-posedness: Statement.
  - Plancherel's identity: Statement and **proof**. (Theorem 3.32)
  - Fourier transform of partial derivatives: Statement. (Proposition 3.35)

- Overview of results & properties (tables)
  - Fourier transform of Schwartz class functions
  - Correspondence between operations in $\mathcal{S}(\mathbb{R}^d)$
  - Important Fourier transforms

4 Problem Sets

It is suggested to understand and learn the following problems from the problem sets

- 1.3
- 2.2
- 3.2 and 3.3
- 4.1.2, 4.1.3 and 4.1.4
- 5.1.4 and 5.1.5
- 6.1.3, 6.1.5 and 6.4
- 8.2
- 9.2 and 9.3
- 10.1.2, 10.1.4, 10.2 and 10.3
- 11.1, 11.2.4 and 11.2.5
- 12.1.5 and 12.2
- 13.1.1, 13.1.4, 13.1.6 and 13.3