Exercise Sheet 1

Exercise 1

Figure 1 is a picture of the unit disk model of the hyperbolic plane $\mathbb{H}^2$. The hyperbolic metric $d_{\mathbb{H}^2}$ on $\mathbb{H}^2$ is such that all the white and blue triangles are congruent (have same area and same side lengths). Argue from the picture that the distance from the midpoint O to some other point P is approximately

$$d_{\mathbb{H}^2}(O,P) \sim \frac{d_{\mathbb{R}^2}(O,P)}{1 - d_{\mathbb{R}^2}(O,P)}$$

where $d_{\mathbb{R}^2}$ is the Euclidean distance.

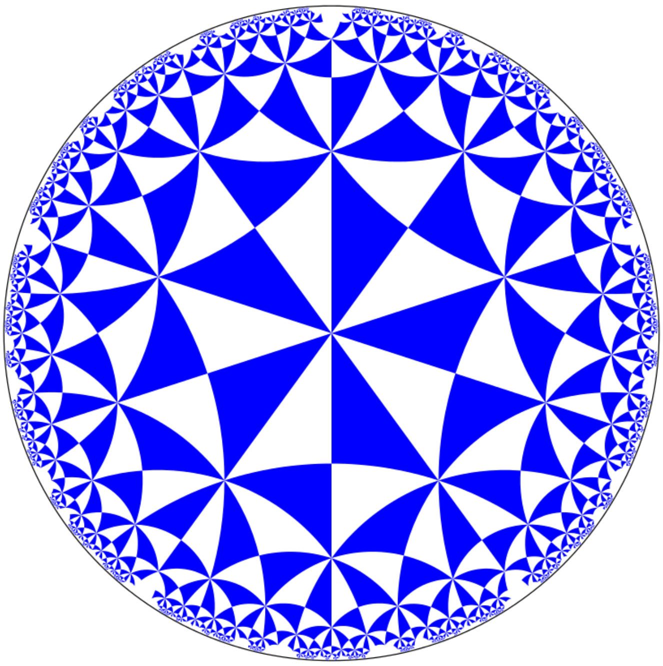

Figure 1: The unit disk $\{x \in \mathbb{R}^2 \colon |x| < 1\}$ with a metric $d_{\mathbb{H}^2}$ is a model for the hyperbolic plane.

Exercise 2

We consider the *extended complex plane* or *Riemann sphere* $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where "the point at infinity" ∞ is just some symbol that is not contained in $\mathbb{C}$. We define the inversion

$$J \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \qquad z \mapsto \begin{cases} \frac{z}{|z|^2} & \text{if } z \in \mathbb{C} \setminus \{0\} \\ \infty & \text{if } z = 0 \\ 0 & \text{if } z = \infty. \end{cases}$$

What is the fixed point set of J? What is $J \circ J$? Is $J|_{\mathbb{C}}$ holomorphic?

Exercise 3

Let $a, b, c, d \in \mathbb{C}$ be complex numbers such that $ad - bc \neq 0$. Then a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$
\begin{aligned}
z &\mapsto \frac{az+b}{cz+d} \quad \text{for } z \in \mathbb{C} \\
\infty &\mapsto \frac{a}{c}
\end{aligned}
$$

is called an *orientation-preserving Möbius transformation*. Here we make the convention that for any constant $c \in \mathbb{C} \setminus \{0\}$, $c/0 = \infty$.

(a) Explain why the condition $ad - bc \neq 0$ is needed.

(b) Show that the composition of two orientation preserving Möbius transformations is again an orientation-preserving Möbius transformation again.

(c) Let Möb_+ be the set of orientation-preserving Möbius transformations equipped with the operation of composition. Show that Möb_+ is a group.

(d) Is J from exercise 2 an orientation preserving Möbius transformation?