Exercise Sheet 10

Exercise 1

(a) Use the cross ratio to determine whether the point $1 + 3i$ lies on the cline defined by the three points $2, 4i, -3 + 5i$.

(b) Suppose a Möbius transformation takes $2, 4, 8$ to $0, 1, \infty$, where does it take i?

(c) Suppose a Möbius transformation takes $0, 1, \infty$ to $2, 4, 8$, where does it take i?

(d) Show that $\mathrm{Im}([z, -i; -1, 1]) > 0$ if and only if $z \in B_1$.

Exercise 2

We consider transformations that preserve the unit disk B_1.

(a) Show that hyperbolic Möbius transformations that preserve the unit disk have two fixed points, both lying on the boundary S^1.

(b) Show that elliptic Möbius transformations that preserve the unit disk have one fixed point, lying in the interior.

(c) Show that parabolic Möbius transformations that preserve the unit disk have one fixed point, lying on the boundary S^1.

Exercise 3

(a) Let A, B, P, Q be points on a line, in this order. Show that $[A, B; P, Q] > 0$.

(b) Let A, B, P, Q be points on a line, in this order. Let X be some point outside the line. Let $\alpha_P, \alpha_Q, \beta_P, \beta_Q$ be the angles at X subtended by AP, AQ, BP and BQ respectively. Use the sine law to show

$$[A, B; P, Q] = \frac{\sin(\alpha_P)\sin(\beta_Q)}{\sin(\alpha_Q)\sin(\beta_P)}.$$

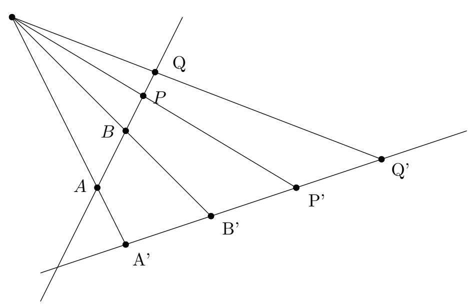

Figure 1: The four points A, B, P, Q on a line are projected to A', B', P', Q' on another line.

(c) Let A, B, P, Q and A', B', P', Q' be points that lie on a line respectively. Show that if they are related by a projection from one line to the other as in Figure 1, that then $[A, B; P, Q] = [A', B'; P', Q']$.

(d) We say that four points $A, B, P, Q \in \mathbb{R}^2$ form a set of *four harmonic points* if their cross ratio satisfy $[A, B; P, Q] = -1$. Verify that given A, B, P on a line, the following ruler-only construction illustrated in Figure 2 can be used to find the fourth harmonic point Q.

Construction: Let X be an arbitrary point outside the line APB. Let Y be an arbitrary point on the line AX. Intersect YB with XP to get W. Intersect AW with BX to get Z. Intersect YZ with APB to get Q.

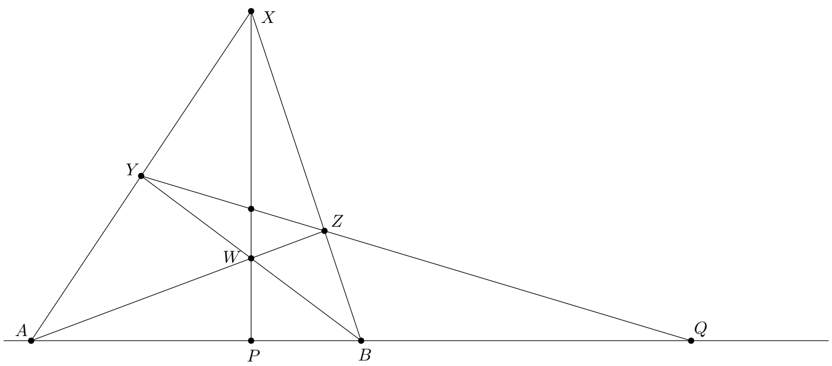

Figure 2: Constructing the fourth harmonic point Q.

Hint: Use projection from X and then projection from W.