Exercise Sheet 11

Exercise 1

Use the hyperbolic version of Geogebra [1] to construct the following hyperbolic objects. Play around with the positions of the points to get a feeling for the geometry. As a challenge, these objects can also be constructed in the usual Geogebra.[2]

(a) Construct 4 hypberolic lines that do not intersect pairwise.

(b) Place two points and construct the perpendicular bisector between them.

(c) Place three points. Find the center of the hyperbolic circle through these points and use the circle tool to draw the circle. Compare the Euclidean and the hyperbolic centerpoints of the circle.

(d) Construct a quadrilateral with three right angles and a fourth angle that is strictly smaller than 90°.

Exercise 2

Which of the following pictures by M.C. Escher are based on hyperbolic geometry?

Figure 1: Pictures by M.C. Escher, Source: `https://mathstat.slu.edu/escher/index.php/Hyperbolic_Geometry_Exercises`.

[1] geogebra.org/classic/tHvDKWdC
[2] www.geogebra.org

Exercise 3

A hyperbolic circle with center $p \in B_1$ and radius $r > 0$ is the set

$$C_{p,r} = \{q \in B_1 \colon d_H(p,q) = r\}.$$

(a) Show that hypberbolic circles are also Euclidean circles.

(b) Show that the Euclidean radius of the hyperbolic circle $C_{0,r}$ is $\tanh(r/2)$.

(c) Let $x \in (0,1)$. Find the hyperbolic center and the hyperbolic radius of the hyperbolic circle containing the three points

$$0, x, \frac{1+i}{2}x \in B_1.$$

Exercise 4

A hyperbolic line is determined by its two points in the boundary $\partial B_1 = S^1$.

(a) Identify the set of hyperbolic lines as quotient of a subset of $S^1 \times S^1$.

(b) Show that the set of hyperbolic lines (with the topology as a quotient of a subset of $S^1 \times S^1$) is homeomorphic to an open Möbius strip.