Exercise Sheet 12

Exercise 1

Sketch the hypberbolic plane and three pairwise intersecting hyperbolic lines. Sketch a fourth hyperbolic line which is ultraparallel to all previous three.

Exercise 2

Let ℓ and ℓ' be two hyperbolic lines that have a common limit point $p \in \partial B_1 = S^1$. Prove that there are sequences of points $x_1, x_2, \ldots \in \ell$ and $y_1, y_2, \ldots \in \ell'$ with $\lim_{n\to\infty}(x_n) = p = \lim_{n\to\infty}(y_n)$, such that there is a constant C with

$$d_H(x_n, y_n) \leq Ce^{-d_H(x_1,x_n)} \qquad \text{for all } n \in \mathbb{N},$$

i.e. the distance between the hyperbolic lines ℓ and ℓ' converges to 0 exponentially fast.

Hint: Use the Taylor expansion of cosh*.*

Exercise 3

(a) Prove that every hyperbolic triangle has angle sum less than 180°.

(b) Show that there are hyperbolic triangles of arbitrarily small positive interior angle sum.

(c) Prove that there is a regular[1] octagon in the hypberbolic plane, all of whose angles are 45°.

Exercise 4

Consider the regular hyperbolic octagon all of whose angles are $2\pi/8$ from Exercise 3(c). Label the sides of the hyperbolic octagon by the letters a, b, a^{-1}, b^{-1}, c, d, c^{-1}, d^{-1} as in Figure 1. Now for each letter in $\{a, b, c, d\}$ glue together the two sides labelled by it and its inverse, respecting the orientation[2]. Denote the resulting object by X.

(a) Prove that all eight vertices of the hyperbolic octagon get identified into one point in X.

(b) Show that for every $x \in X$, we can identify a neighborhood of x with a neighborhood of a point in the hyperbolic plane. This way we can give X a local hyperbolic metric, we then say that X is a *hyperbolic surface.*

(c) Show that X is homeomorphic to a double torus.

[1] An n-gon is regular if all its sides have the same lengths and the angles at all vertices are the same.

[2] This construction is analogous to the construction of a torus by identifying the opposite sides of a square.

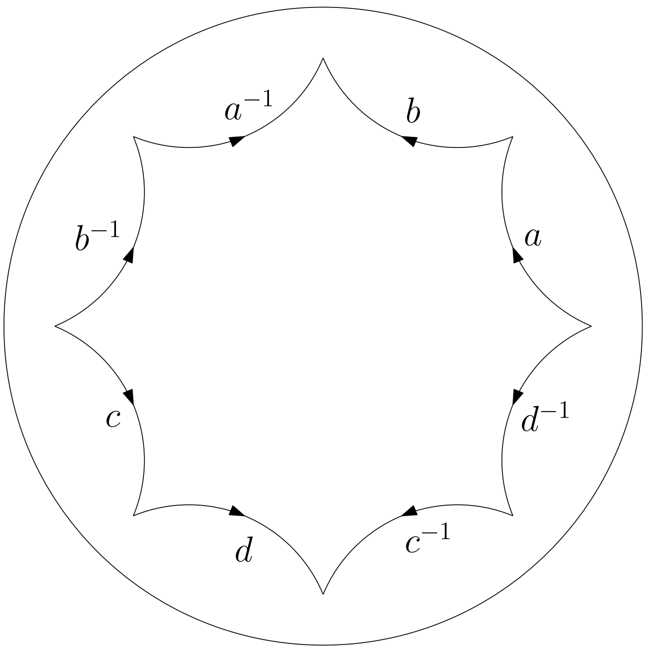

Figure 1: A hyperbolic octagon whose sides are to be identified.