Exercise Sheet 2

Exercise 1

Identify $\mathbb{R}^2 \cong \mathbb{C}$.

(a) Show that the inverse function

$$\begin{aligned} \mathbb{C}\setminus\{0\} &\to \mathbb{C}\setminus\{0\} \\ z &\mapsto 1/z \end{aligned}$$

is an orientation-preserving map.

(b) Define $I\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by extending the inverse to have suitable values at 0 and at ∞. Prove that I is a homeomorphism of $\hat{\mathbb{C}}$ with respect to the topology defined in class.

Exercise 2

Consider the three Möbius transformations $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$I\colon z \mapsto \frac{1}{z} \qquad J\colon z \mapsto \frac{1}{\bar{z}} \qquad C\colon z \mapsto \bar{z}.$$

(a) Describe the group generated by I, J and C.

(b) Describe the actions of I, J and C on the Riemann sphere considered as the round sphere S^2.

(c) Which of these maps are orientation-preserving? Which are orientation-reversing?

Exercise 3

(a) Give an example of a real affine map $\mathbb{R}^2 \to \mathbb{R}^2$ that is not a similarity.

(b) Classify the similarities of $\mathbb{R}^2$ in terms of their fixed points.

(c) Show that the group of similarities of the plane $\mathrm{Sim}(\mathbb{R}^2)$ has the structure of a semidirect product $\mathrm{Sim}(\mathbb{R}^2) = \mathrm{Sim}_+(\mathbb{R}^2) \rtimes \mathbb{Z}/2\,\mathbb{Z}$.