Exercise Sheet 3

Exercise 1

(a) Show that orientation-preserving Möbius transformations $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ have exactly either one or two fixed points, or are the identity.

(b) Give examples of orientation-preserving Möbiustransformations with one (resp. two) fixed points.

(c) Draw a picture of how the Möbiustransformations from (b) act on $\hat{\mathbb{C}}$.

(d) Does the statement of (a) also hold for orientation-reversing Möbius transformations?

Exercise 2

(a) Show that the group of Möbius transformations Möb acts transitively on $\hat{\mathbb{C}}$, i.e. for any $x, y \in \hat{\mathbb{C}}$ there is a $g \in$ Möb such that $g(x) = y$.

(b) Determine all the combinations of coefficients a, b, c, d such that the corresponding Möbiustransformation is the identity.

(c) Find all orientation-preserving Mobius transformations $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that send

$$\infty \mapsto i \mapsto 0 \mapsto -i \mapsto \infty.$$

(d) What are the possible values for $f(1)$ under a Mobius transformation as in (c)?

Exercise 3

Answer the questions (a) - (e) for the functions (**1**) - (**7**). All functions are continuous functions of the Riemann-sphere $\hat{\mathbb{C}}$. For (**1**) - (**5**) we have $f(\infty) = \infty$.

(a) Find the fixed point set $\{z \in \hat{\mathbb{C}} \colon f_i(z) = z\}$.

(b) Does f_i preserve the unit disk?

(c) Is f_i an affine map, when restricted to $\mathbb{C} \cong \mathbb{R}^2$?

(d) Is f_i a similarity, when restricted to $\mathbb{C}$?

(e) If f_i orientation-preserving or orientation-reversing?

(**1**) $f_1 \colon z \mapsto z + a$, where $a \in \mathbb{C}$ fixed.

(**2**) $f_2 \colon z \mapsto rz$, where $r > 0$ fixed.

(**3**) $f_3 \colon z \mapsto e^{i\varphi} z$, where $\varphi \in [0, 2\pi)$ fixed.

(**4**) $f_4 \colon z \mapsto \bar{z}$.

(**5**) $f_5 \colon z \mapsto z^2$.

(**6**) $f_6 \colon z \mapsto \frac{z}{|z|^2}$, where $f(\infty) = 0$ and $f(0) = \infty$.

(**7**) $f_7 \colon z \mapsto \frac{1}{z}$, where $f(\infty) = 0$ and $f(0) = \infty$.