Exercise Sheet 4

Exercise 1

For the following exercises, first reason geometrically and then find an algebraic description.

(a) Let $p \in \mathbb{C}$. Describe the point-reflection $Q_p\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

(b) Show that $Q_p^2 = \mathrm{id}$.

(c) For $p, q \in \mathbb{C}$, what is $Q_p \circ Q_q$?

(d) Let $L \subseteq \mathbb{C}$ be a line. Describe the reflection $R_L\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ along L.

(e) Show that for every line L through 0, R_L can be written as $z \mapsto e^{i\varphi}\bar{z}$ for some $\varphi \in \mathbb{R}$.

(f) What is the line along which $z \mapsto e^{i\varphi}\bar{z}$ reflects?

(g) Write Q_p and R_L as Möbius transformations.

Exercise 2

Let $b \in \mathbb{C}$ and $\lambda > 0$. Let T_b be the translation by b and let M_λ be the multiplication by λ.

(a) Describe the effect of $T_b \circ M_\lambda \circ T_{-b}$ geometrically.

(b) Describe the effect of $M_\lambda \circ T_b \circ M_{\lambda^{-1}}$ geometrically.

(c) Express the transformations in (a) and (b) as Möbius transformations.

Exercise 3

Given a group of matrices $\mathrm{G} \subseteq M^{n\times n}(\mathbb{C})$, let $Z_{\mathrm{G}} = \{g \in G\colon \forall h \in G\colon gh = hg\}$ be the center. Then PG is defined to be the group $\mathrm{G}/Z_{\mathrm{G}}$. Recall that for any field F and any $n \geq 1$,

$$\mathrm{GL}(n, F) = \{g \in M^{n\times n}(F)\colon \det(g) \neq 0\}$$
$$\mathrm{SL}(n, F) = \{g \in M^{n\times n}(F)\colon \det(g) = 1\}.$$

(a) Show that $\mathrm{PGL}(2, \mathbb{R}) \not\cong \mathrm{PSL}(2, \mathbb{R})$.

(b) Show that $\mathrm{PGL}(3, \mathbb{R}) \cong \mathrm{PSL}(3, \mathbb{R})$.

(c) What about $\mathrm{PGL}(n, \mathbb{R})$ and $\mathrm{PSL}(n, \mathbb{R})$ for $n \geq 4$?

(d) Show that $\mathrm{PGL}(n, \mathbb{C}) \cong \mathrm{PSL}(n, \mathbb{C})$ for all $n \geq 2$.