Exercise Sheet 7

Exercise 1

For $-1 < t < 1$, consider the Apollonian slide $K_t \colon B_1 \to B_1$ defined by

$$K_t(z) = \frac{z+t}{tz+1} \qquad z \in B_1.$$

(a) Find a t' such that $K_{t'} = K_t \circ K_t$.

(b) A *one-parameter subgroup* in a group G is a function $h \colon \mathbb{R} \to G$, such that

$$h(s+t) = h(s)h(t) \qquad s,t \in \mathbb{R}.$$

Reparametrize the family K_t as a new family $\tilde{K}_t$ which is a one parameter subgroup in $\text{Möb}(B_1)$.

You may want to look up the addition theorems for hyperbolic trigonometric functions.

Exercise 2

In class, we proved that Möbius transformations on S^2 are conformal by proving it first for affine maps, and then using a generating set of Möb. This exercise investigates an algernative proof using charts.

Let f be a Möbius transformation and $\tilde{f} = \sigma^{-1} \circ f \circ \sigma$ its corresponding map on S^2. We write $\sigma \colon S^2 \to \hat{\mathbb{C}}$ for the stereographic projection centered at the north pole and $\sigma' \colon S^2 \to \hat{\mathbb{C}}$ for the stereographic projection centered at the south pole.

Since $f|_{\mathbb{C}\setminus\{f^{-1}(\infty)\}}$ is holomorphic on $\mathbb{C} \setminus \{f^{-1}(\infty)\}$ and σ is conformal, $\tilde{f}$ is conformal at all points $p \in S^2 \setminus \{N, \tilde{f}^{-1}(N)\}$. We want to show that $\tilde{f}$ is conformal everywhere.

(a) Consider the four maps of $\hat{\mathbb{C}}$

$$\sigma \circ \tilde{f} \circ \sigma^{-1}, \quad \sigma' \circ \tilde{f} \circ (\sigma')^{-1}, \quad \sigma \circ \tilde{f} \circ (\sigma')^{-1}, \quad \sigma' \circ \tilde{f} \circ \sigma^{-1}.$$

Show that for any point $p \in S^2$, at least one of these four maps can be used to prove that $\tilde{f}$ is conformal at p. In this formulation, we don't have to use a generating set.

(b) Why isn't it enough just to consider

$$\sigma \circ \tilde{f} \circ \sigma^{-1}, \quad \sigma' \circ \tilde{f} \circ (\sigma')^{-1}?$$

(c) If we take the approach in (b), what is the set of Möbius transformations for which the proof fails?