Exercise Sheet 9

Exercise 1

Let $s \leq t \in (-1,1)$ and define

$$\mathrm{cr}(s,t) := \mathrm{cr}(t,s) := \frac{(t+1)(1-s)}{(s+1)(1-t)}.$$

(a) Explain how cr is related to the cross ratio $[z_1, z_2; z_3, z_4]$ in class.

(b) Show that $\mathrm{cr}(s,t) \geq 1$ and $\mathrm{cr}(s,t) = 1$ if and only if $t = s$.

(c) Show that for all $s, t, r \in (-1,1)$, $\mathrm{cr}(s,r) \leq \mathrm{cr}(s,t)\,\mathrm{cr}(t,r)$.

(d) In view of (a) and (b), how could the cross-ratio cr be used to define a metric on the real interval $(-1,1)$.

(e) Check that the Apollonian slide $K_t \colon z \mapsto \frac{z+t}{tz+1}$ for $t \in (-1,1)$ is an isometry of $(-1,1)$ with the distance from (d), meaning $\forall x, y \in (-1,1), d(x,y) = d(K_t(x), K_t(y))$.

The metric defined on the subset of B_1 will be expanded to the hyperbolic metric on B_1.

Exercise 2

(a) Identify $\mathrm{Möb}(B_1)$ with the set of all injective maps of the set $\{0,1,2\}$ into S^1.

(b) Show that $\mathrm{Möb}(B_1)$ is homeomorphic to an open subset of the 3-torus $S^1 \times S^1 \times S^1$. What set is excluded?

(c) The 3-torus has the advantage that you can visualize it. It is a cube with its sides suitably identified. Try to draw a picture of the topology of $\mathrm{Möb}(B_1)$.

(d) Is $\mathrm{Möb}(B_1)$ connected? Is $\mathrm{PSL}(2,\mathbb{R})$ connected? Are they simply connected?