# Solutions 1

## Exercise 1

Figure 1 is a picture of the unit disk model of the hyperbolic plane $\mathbb{H}^2$. The hyperbolic metric $d_{\mathbb{H}^2}$ on $\mathbb{H}^2$ is such that all the white and blue triangles are congruent (have same area and same side lengths). Argue from the picture that the distance from the midpoint $O$ to some other point $P$ is approximately

$$d_{\mathbb{H}^2}(O,P) \sim \frac{d_{\mathbb{R}^2}(O,P)}{1 - d_{\mathbb{R}^2}(O,P)}$$

where $d_{\mathbb{R}^2}$ is the Euclidean distance.

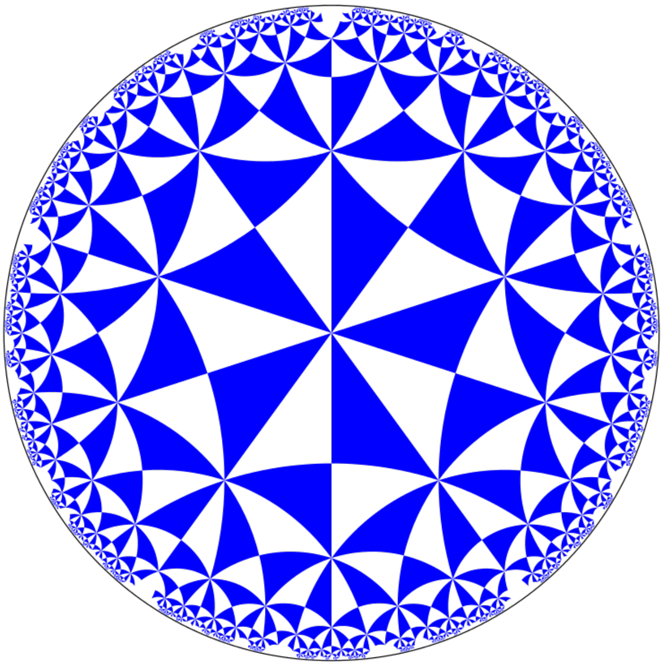

Figure 1: The unit disk $\{x \in \mathbb{R}^2 : |x| < 1\}$ with a metric $d_{\mathbb{H}^2}$ is a model for the hyperbolic plane.

**Solution:**
We first note that the hyperbolic distance looks invariant under rotation around $O$, hence $d_{\mathbb{H}^2}(O,P)$ only depends on $|P|$.

We note that the triangles seem to get smaller as we go towards the boundary. Even though not infinitely many triangles can be drawn, we may assume that there are infinitely many triangles. This also follows from homogeneity (the space looks the same around every point). From this we conclude that the hyperbolic distance $d_{\mathbb{H}^2}(O,P)$ has to go towards $\infty$ as

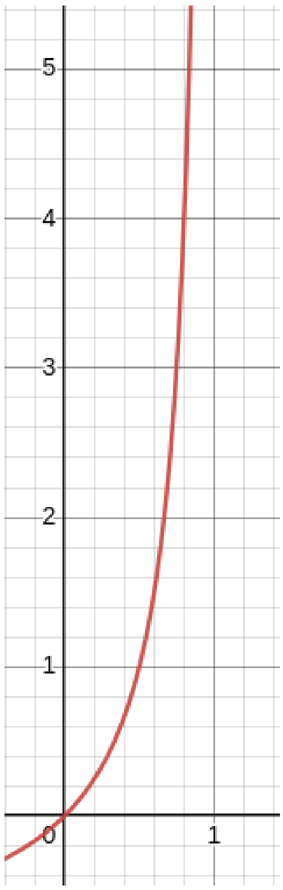


Figure 2: The function $x \mapsto x/(1-x)$ on the interval $[0,1]$.

$|P| \to 1$. We also know $d_{\mathbb{H}^2}(O,O) = 0$. The function $x \mapsto x/(1-x)$ (cf. Figure 2) satsifies these properties and is hence a good candidate.

Actually measuring a few points with a ruler will give you more evidence that this formula holds.

## Exercise 2

We consider the *extended complex plane* or *Riemann sphere* $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where "the point at infinity" $\infty$ is just some symbol that is not contained in $\mathbb{C}$. We define the inversion

$$J \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \qquad z \mapsto \begin{cases} \frac{z}{|z|^2} & \text{if } z \in \mathbb{C} \setminus \{0\} \\ \infty & \text{if } z = 0 \\ 0 & \text{if } z = \infty. \end{cases}$$

What is the fixed point set of $J$? What is $J \circ J$? Is $J|_{\mathbb{C}}$ holomorphic?

**Solution:**
For $z \in \mathbb{C} \setminus \{0\}$, we have $z = z/|z|^2$ if and only if $|z|^2 = 1$, i.e. $|z| = 1$. As $J$ swaps 0 and $\infty$, the set of fixed points of $J$ is the unit circle.

For $z \in \mathbb{C} \setminus \{0\}$, we have

$$J(J(z)) = \frac{\frac{z}{|z|^2}}{\left|\frac{z}{|z|^2}\right|^2} = \frac{z|z|^4}{|z|^2|z|^2} = z,$$

so $J \circ J$ is the identity function on $\hat{\mathbb{C}}$.

A function $f = u + iv \colon \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if $u, v \colon \mathbb{C} \to \mathbb{R}$ are differentiable (as real functions $\mathbb{R}^2 \to \mathbb{R}$) and they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $z = x + iy \in \mathbb{C}$.

For $z = x + iy \in \mathbb{C}$ we have

$$J(z) = \frac{z}{|z|^2} = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2} = u(z) + iv(z).$$

The real functions $u$ and $v$ are differentiable, but

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \neq \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}$$

so the Cauchy-Riemann equations are not satisfied and $J|_{\mathbb{C}}$ is not holomorphic.

## Exercise 3

Let $a, b, c, d \in \mathbb{C}$ be complex numbers such that $ad - bc \neq 0$. Then a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$\begin{aligned} z &\mapsto \frac{az + b}{cz + d} \quad \text{for } z \in \mathbb{C} \\ \infty &\mapsto \frac{a}{c} \end{aligned}$$

is called an *orientation-preserving Möbius transformation*. Here we make the convention that for any constant $c \in \mathbb{C} \setminus \{0\}$, $c/0 = \infty$.

(a) Explain why the condition $ad - bc \neq 0$ is needed.

(b) Show that the composition of two orientation preserving Möbius transformations is again an orientation-preserving Möbius transformation again.

(c) Let $\text{Möb}_+$ be the set of orientation-preserving Möbius transformations equipped with the operation of composition. Show that $\text{Möb}_+$ is a group.

(d) Is $J$ from exercise 2 an orientation preserving Möbius transformation?

**Solution:**

(a) A problem that can arise when defining a function like this is the case $0/0$. In fact if

$$\begin{aligned} az + b &= 0 \\ cz + d &= 0 \end{aligned}$$

then $ad - bc = 0$. To see this let's first assume $a \neq 0$:

If the two linear equations are zero, then $z = -b/a$, hence $c(-b/a) + d = 0$, which is equivalent to $ad - bc = 0$.

If however $a = 0$ and $c \neq 0$, then we conclude $ad - bc = 0$ similarly. Finally if both $a = 0$ and $c = 0$, then $ad - bc = 0$ trivially.

In all cases it follows that if $ad - bc \neq 0$, then the formula of the Möbiustransformation is not of the form $0/0$.

(b) Let $a, b, c, d, a', b', c', d' \in \mathbb{C}$ with $ad - bc \neq 0 \neq a'd' - b'c'$. We have

$$\begin{aligned} \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} &= \frac{aa'z + ab' + b(c'z + d')}{c'z + d'} \cdot \frac{c'z + d'}{ca'z + cb' + d(c'z + d')} = \\ &= \frac{(aa' + bc')z + ab' + bd'}{(ca' + dc')z + cb' + dd'} \end{aligned}$$

which is a Möbius transformation with coefficients $aa' + bc', ab' + bd', ca' + dc', cb' + dd'$. We have to check the condition $ad - bc \neq 0$:

$$\begin{aligned} &(aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc') \\ &\quad = aca'b' + ada'd' + bcc'b' + bdc'd' - acb'a' - adb'c' - bcd'a' - bdd'c' \\ &\quad = ad(a'd' - b'c') + bc(c'b' - d'a') = (ad - bc)(a'd' - b'c') \neq 0. \end{aligned}$$

(c) We have to check the group laws. By definition of composition, associativity holds. The identity map is a Möbius transformation with $a = 1, b = 0, c = 0$ and $d = 1$ and serves as the neutral element. It remains to show that every Möbius transformation has an inverse. For this we use the following equivalences

$$\begin{aligned} \frac{az + b}{cz + d} &= w \\ az + b &= wcz + wd \\ (a - wc)z &= wd - b \\ z &= \frac{dw - b}{-cw + a}, \end{aligned}$$

which is a Möbius transformation with coefficients $d, -b, -c, a$. The condition $ad - bc = 0$ remains unchanged.

**It may be a good exercise to try to translate everything in terms of matrices.**

(d) The answer is no, if it were, then we would get that $a/c = 0$, hence $a = 0$ from $J(\infty) = 0$ and that $b/d = \infty$, hence $d = 0$ from $J(0) = \infty$. $J(1) = 1$ would then imply that $b/c = 1$, hence $b = c$, and thus determine the Möbius transformation to be $f\colon z \mapsto 1/z$. However $f(i) = -i$, while $J(i) = i$, whence $J$ is not a Möbius transformation.