Solutions 10

Exercise 1

(a) Use the cross ratio to determine whether the point $1+3i$ lies on the cline defined by the three points $2, 4i, -3+5i$.

(b) Suppose a Möbius transformation takes $2, 4, 8$ to $0, 1, \infty$, where does it take i?

(c) Suppose a Möbius transformation takes $0, 1, \infty$ to $2, 4, 8$, where does it take i?

(d) Show that $\mathrm{Im}([z, -i; -1, 1]) > 0$ if and only if $z \in B_1$.

Solution:

(a) The point lies on the cline if and only if the cross ratio is a real number.

$$\begin{aligned}[4i, 2; -3+5i, 1+3i] &= \frac{(4i-(-3+5i))(2-(1+3i))}{(2-(-3+5i))(4i-(1+3i))} \\ &= \frac{(3-i)(1-3i)}{(5-5i)(-1+i)} = \frac{-10i}{10i} = -1 \in \mathbb{R},\end{aligned}$$

so yes, the point $1+3i$ lies on the cline.

(b) We saw in the lecture that the Möbius transformation

$$f\colon z \mapsto \frac{(z-2)(4-8)}{(z-8)(4-2)} = \frac{-2z+4}{z-8} \qquad (= [z, 4; 2, 8] \text{ when } z \notin \{2, 4, 8\}),$$

sends $2 \mapsto 0, 4 \mapsto 1, 8 \mapsto \infty$ as required. Now

$$\begin{aligned}f(i) &= \frac{-2i+4}{i-8} = \frac{-2i+4}{i-8}\,\frac{i+8}{i+8} \\ &= \frac{32+2+i(4-16)}{-1-64} = \frac{-34+12i}{65}\end{aligned}$$

(c) We want the inverse Möbius transformation to the one in (b). Interpreting f as a matrix in $\mathrm{PSL}(2, \mathbb{C})$

$$f^{-1} = \begin{bmatrix} -2 & 4 \\ 1 & -8 \end{bmatrix}^{-1} = \frac{1}{12}\begin{bmatrix} -8 & -4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 1 & 2 \end{bmatrix}$$

and thus

$$\begin{aligned}f^{-1}(i) &= \frac{8i+4}{i+2} = \frac{8i+4}{i+2}\,\frac{i-2}{i-2} \\ &= \frac{-8-8+i(4-16)}{-1-4} = \frac{16+12i}{5}\end{aligned}$$

(d) We recall that for $z \notin \{0, 1, \infty\}$, $z = [z, 1; 0, \infty]$. In particular, z is lies in the upper half plane $H_+ = \{a + ib \colon b > 0\}$ if and only if $\mathrm{Im}(z) = \mathrm{Im}([z, 1; 0, \infty]) > 0$.

By triple transitivity we know that there is a transformation f that sends the upper half plane to the unit disk B_1 by sending $1 \mapsto -i, 0 \mapsto -1, \infty \mapsto 1$ (Cayley-transformation). Since $f(H_+) = B_1$, we have $z \in B_1$ if and only if

$$f^{-1}(z) \in H_+,$$

meaning

$$\mathrm{Im}([f^{-1}(z), 1; 0, \infty]) > 0.$$

Since f^{-1} is a Möbius transformation, f^{-1} preserves the cross ratio, hence

$$\mathrm{Im}([z, -i; -1, 1]) = \mathrm{Im}([f^{-1}(z), 1; 0, \infty]) > 0.$$

Exercise 2

We consider transformations that preserve the unit disk B_1.

(a) Show that hyperbolic Möbius transformations that preserve the unit disk have two fixed points, both lying on the boundary S^1.

(b) Show that elliptic Möbius transformations that preserve the unit disk have one fixed point, lying in the interior.

(c) Show that parabolic Möbius transformations that preserve the unit disk have one fixed point, lying on the boundary S^1.

Solution:

We note that for all Möbius transformations $f, r \in$ Möb, the fixed point sets satisfy

$$\mathrm{Fix}_{\hat{\mathbb{C}}}(r \circ f \circ r^{-1}) = r\left(\mathrm{Fix}_{\hat{\mathbb{C}}}(f)\right).$$

(a) By definition, a hyperbolic Möbius transformation is one that is conjugated to $M_a \colon z \mapsto az$ for $a > 0$. The only clines that are preserved by M_a are the straight lines through the two fixed points 0 and ∞. If $h = r \circ M_a \circ r^{-1}$ is a Möbius transformation that preserves the unit disk, by continuity it also has to preserve its boundary S^1, which is a cline. Möbius transformations such as r^{-1} send clines to clines, thus $r^{-1}(S^1)$ is a cline preserved by M_a, i.e. a straight line through 0 and ∞. This means in particular that $r(\{0, \infty\}) \subseteq S^1$. Then

$$\mathrm{Fix}_{\hat{\mathbb{C}}}(h) = \mathrm{Fix}_{\hat{\mathbb{C}}}(r \circ M_a \circ r^{-1}) = r\left(\mathrm{Fix}_{\hat{\mathbb{C}}}(M_a)\right) = \{r(0), r(\infty)\} \subseteq S^1$$

i.e. h has exactly two fixed points, both lying on the boundary S^1.

(b) Every elliptic Möbius transformation is of the form $r \circ M_\varphi \circ r^{-1}$, where $M_\varphi(z) = ze^{i\varphi}$ for $\varphi \in (0, 2\pi)$. We notice that the only clines that are preserved under M_φ (and whose two connected components are also preserved) are the circles with center 0. (Remark: When $\varphi \neq \pi$, the second condition on the connected component is not necessary.)

If an elliptic Möbius transformation $e = r \circ M_\varphi \circ r^{-1}$ preserves B_1, then it also preserves the boundary S_1 (and it preserves the connected components of $\hat{\mathbb{C}}$). Thus $r^{-1}(S^1)$ is a cline preserved by M_φ (and its connected components are also preserved by M_φ). Thus $r^{-1}(S^1)$ is a circle with center 0 and contains no fixed points of M_φ on the circle, and exactly one fixed point, namely either 0 or ∞ in each of the connected components. We have

$$\mathrm{Fix}_{\hat{\mathbb{C}}}(e) = \mathrm{Fix}_{\hat{\mathbb{C}}}(r \circ M_\varphi \circ r^{-1}) = r\left(\mathrm{Fix}_{\hat{\mathbb{C}}}(M_\varphi)\right) = \{r(0), r(\infty)\}.$$

Since 0 and ∞ are in different connected components defined by the circle $r^{-1}(S^1)$, the points $r(0)$ and $r(\infty)$ are in different components defined by the circle S^1, i.e. exactly one of $r(0)$ and $r(\infty)$ is inside B_1. We conclude that every elliptic Möbius transformation that preserves the unit disk has exactly one fixed point, somewhere in the interior of B_1.

(c) A Möbius transformation is called parabolic if it is conjugated to $T_a\colon z \mapsto z + a$ with $a \in \mathbb{R}$. The only clines preserved by T_a are the horizontal straight lines, whose only fixed point is ∞. Let $p = r \circ T_a \circ r^{-1}$ be a parabolic Möbius transformation preserving B_1. Hence p also preserves S^1, thus $r^{-1}(S^1)$ is a cline preserved by T_a. This means that T_a has exactly one fixed point, which lies in $r^{-1}(S^1)$. Thus

$$\mathrm{Fix}_{\hat{\mathbb{C}}}(p) = \mathrm{Fix}_{\hat{\mathbb{C}}}(r \circ T_a \circ r^{-1}) = r\left(\mathrm{Fix}_{\hat{\mathbb{C}}}(T_a)\right) = \{r(\infty)\}$$

and the only fixed point of p is $r(\infty) \in S^1$.

Exercise 3

(a) Let A, B, P, Q be points on a line, in this order. Show that $[A, B; P, Q] > 0$.

(b) Let A, B, P, Q be points on a line, in this order. Let X be some point outside the line. Let $\alpha_P, \alpha_Q, \beta_P, \beta_Q$ be the angles at X subtended by AP, AQ, BP and BQ respectively. Use the sine law to show

$$[A, B; P, Q] = \frac{\sin(\alpha_P)\sin(\beta_Q)}{\sin(\alpha_Q)\sin(\beta_P)}.$$

(c) Let A, B, P, Q and A', B', P', Q' be points that lie on a line respectively. Show that if they are related by a projection from one line to the other as in Figure 1, that then $[A, B; P, Q] = [A', B'; P', Q']$.

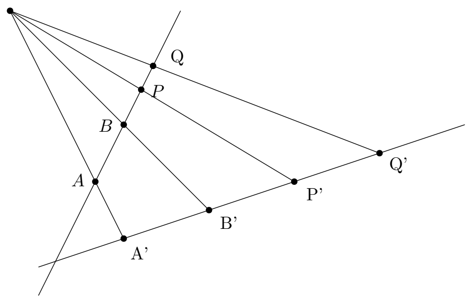

Figure 1: The four points A, B, P, Q on a line are projected to A', B', P', Q' on another line.

(d) We say that four points $A, B, P, Q \in \mathbb{R}^2$ form a set of *four harmonic points* if their cross ratio satisfy $[A, B; P, Q] = -1$. Verify that given A, B, P on a line, the following ruler-only construction illustrated in Figure 2 can be used to find the fourth harmonic point Q.

Construction: Let X be an arbitrary point outside the line APB. Let Y be an arbitrary point on the line AX. Intersect YB with XP to get W. Intersect AW with BX to get Z. Intersect YZ with APB to get Q.

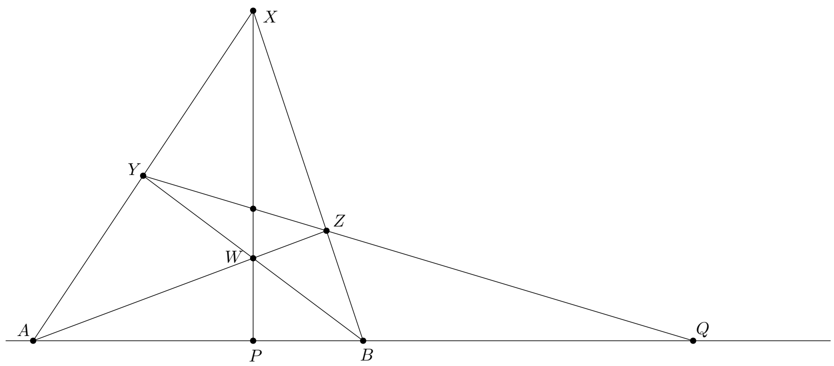

Figure 2: Constructing the fourth harmonic point Q.

Hint: Use projection from X and then projection from W.

Solution:

(a) We recall that

$$[A, B; P, Q] = \frac{(A-P)(B-Q)}{(A-Q)(B-P)} \in \mathbb{R}$$

since A, B, P, Q lie on a common cline. By tripple transitivity of Möbius transformations we can find $f \in$ Möb such that $f(A) = 0, f(B) = 1, f(Q) = \infty$ (and then automatically $f(P) \in (1, \infty)$, since

P lies on the beween B and Q on the cline). Since f preserves the cross ratio, we may assume $A = 0, B = 1, P \in (1, \infty), Q = \infty$.

Recall that when $z \notin \{0, 1, \infty\}$, then $z = [z, 1; 0, \infty]$. Assuming $z \in (1, \infty)$, we have that $[z, 1; 0, \infty] > 1$. By a formula from the lecture, (or an elementary calculation), we permute the entries to obtain

$$[0, 1; z, \infty] = \frac{z}{z-1}$$

which shows that $[A, B; P, Q] > 0$ since $z > 1$.

(b) Since $[A, B; P, Q] > 0$ we have

$$[A, B; P, Q] = |[A, B; P, Q]| = \frac{|A-P| \cdot |B-Q|}{|A-Q| \cdot |B-P|}.$$

Let α be the angle at A subtending QX and let β be the angle at B subtending QX. Then we apply the sine law four times to obtain

$$\frac{\sin(\alpha_P)}{|A-P|} = \frac{\sin(\alpha)}{|X-P|}$$
$$\frac{\sin(\alpha_Q)}{|A-Q|} = \frac{\sin(\alpha)}{|X-Q|}$$
$$\frac{\sin(\beta_P)}{|B-P|} = \frac{\sin(\beta)}{|X-P|}$$
$$\frac{\sin(\beta_Q)}{|B-Q|} = \frac{\sin(\beta)}{|X-Q|}.$$

We plug in

$$[A, B; P, Q] = \frac{\frac{\sin(\alpha_P)|X-P|}{\sin(\alpha)} \frac{\sin(\beta_Q)|X-Q|}{\sin(\beta)}}{\frac{\sin(\alpha_Q)|X-Q|}{\sin(\alpha)} \frac{\sin(\beta_P)|X-P|}{\sin(\beta)}} = \frac{\sin(\alpha_P)\sin(\beta_Q)}{\sin(\alpha_Q)\sin(\beta_P)}.$$

(c) Up to relabelling points, we are in the situation of (b), in which whe can see that the cross ratio does not depend directly on the line, the angles are enough. Since for any two lines as in Figure 1, the angles $\alpha_P, \alpha_Q, \beta_P, \beta_Q$ are the same, their cross ratios also have to be the same.

(d) Let T denote the intersection $XP \cap YZ$. Projecting from X gives us

$$[A, B; P, Q] = [Y, Z; T, Q].$$

Projecting from W gives

$$[A, B; P, Q] = [Z, Y; T, Q].$$

Permuting the first two entries of the cross ratio has the effect of inverting the cross ratio, as was seen in the lecture or follows from a short calculation. Thus we have

$$[A, B; P, Q] = [A, B; P, Q]^{-1},$$

hence $[A, B; P, Q]^2 = 1$ and thus $[A, B; P, Q] \in \{\pm 1\}$. Since the cross ratio can never reach 1 it has to be $[A, B; P, Q] = -1$, hence the four points are harmonic.