# Solutions 11

## Exercise 1

Use the hyperbolic version of Geogebra [1] to construct the following hyperbolic objects. Play around with the positions of the points to get a feeling for the geometry. As a challenge, these objects can also be constructed in the usual Geogebra.[2]

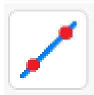
(a) Construct 4 hypberolic lines that do not intersect pairwise.

(b) Place two points and construct the perpendicular bisector between them.

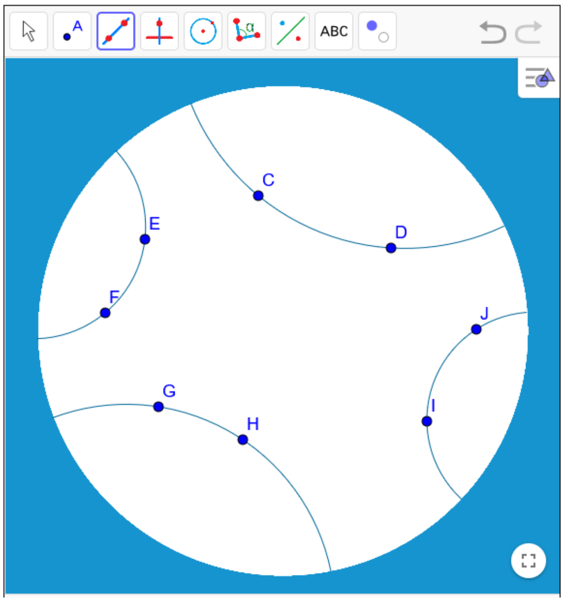
(c) Place three points. Find the center of the hyperbolic circle through these points and use the circle tool to draw the circle. Compare the Euclidean and the hyperbolic centerpoints of the circle.

(d) Construct a quadrilateral with three right angles and a fourth angle that is strictly smaller than 90°.

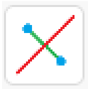
**Solution:**

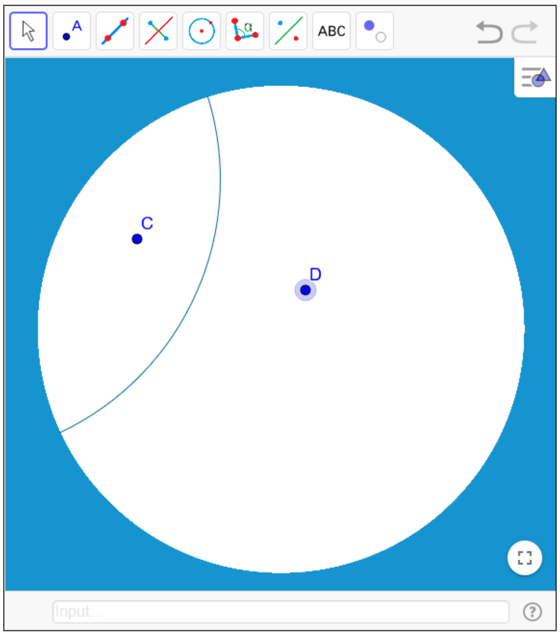
(a) We can use the geodesics tool. The icon for it looks like this: .

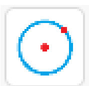
We can place the points near the boundary, so that we have enough place for the four hyperbolic lines.

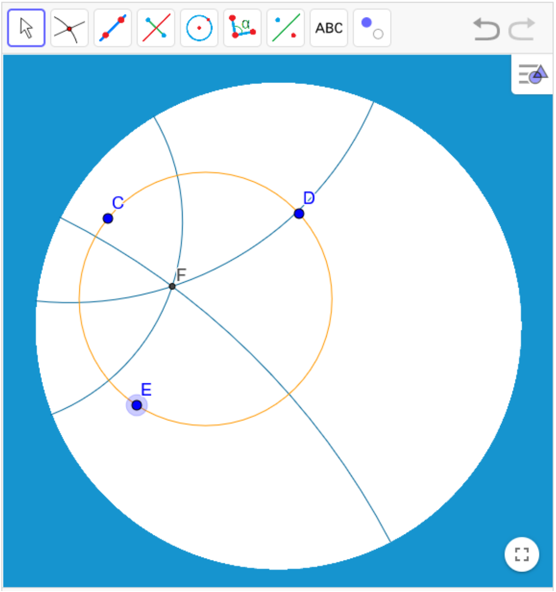



[1] geogebra.org/classic/tHvDKWdC
[2] www.geogebra.org

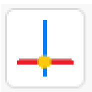
(b) There is a tool for perpendicular bisectors: , but one could also construct it differently. The picture then looks like this:

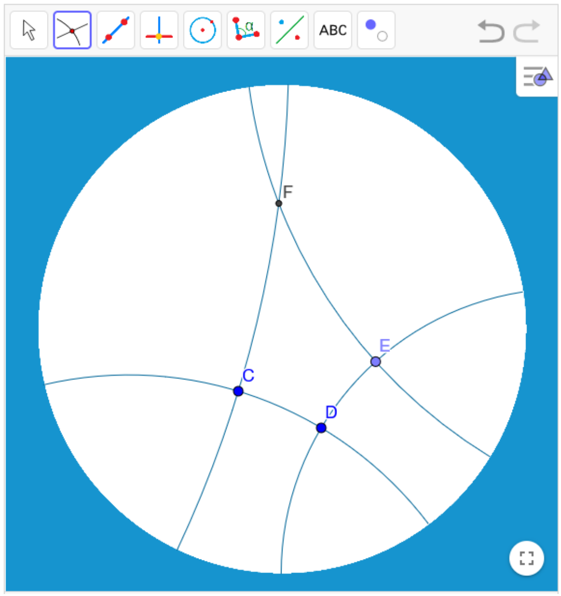



(c) To find the midpoint of the circle, we can construct two perpendicular bisectors. These intersect in the midpoint of the circle. We use the circle tool  to draw the circle.




We notice that from our Euclidean perspective, the hyperbolic midpoint is not in the middle of the circle. This makes intuitive sense, since distances increase as we move towards the boundary.

(d) There is a useful tool , that draws the perpendicular to a line at a point. We can use it to construct right angles. As long as the last two sides intersect, the last angle is always automatically smaller than $\pi/2$.




## Exercise 2

Which of the following pictures by M.C. Escher are based on hyperbolic geometry?

Figure 1: Pictures by M.C. Escher, Source: https://mathstat.slu.edu/escher/index.php/Hyperbolic_Geometry_Exercises.

**Solution:**
It is 1, 4 and 6. In these pictures all the repeating patterns have the same hyperbolic size.

## Exercise 3

A hyperbolic circle with center $p \in B_1$ and radius $r > 0$ is the set

$$C_{p,r} = \{q \in B_1 \colon d_H(p,q) = r\}.$$

(a) Show that hypberbolic circles are also Euclidean circles.

(b) Show that the Euclidean radius of the hyperbolic circle $C_{0,r}$ is $\tanh(r/2)$.

(c) Let $x \in (0,1)$. Find the hyperbolic center and the hyperbolic radius of the hyperbolic circle containing the three points

$$0, x, \frac{1+i}{2}x \in B_1.$$

**Solution:**

(a) When the center is $p = 0$, the hyperbolic circles are Euclidean circles, as the hyperbolic distance is invariant under rotation around 0. By transitivity of $\text{Möb}(B_1)$ on the hyperbolic plane, we can find for every center $p \in B_1$ a transformation $f_p$ with $f_p(0) = p$. Then $C_{p,r} = f_p(C_{0,r})$. Since $f_p$ sends clines to clines, $f_p(C_{0,r}) = C_{p,r}$ has to be a cline. It cannot be a line, since it is contained in $B_1$, hence it has to be a circle.

(b) Let $x \in (0,1) \cap C_{0,r}$ be the unique point on the circle and on the positive real axis. We have

$$r = d_{B_1}(0,x) = \log([0,x;1,-1]) = \log\left(\frac{(-1)\cdot(x+1)}{1\cdot(x-1)}\right)$$

We resolve

$$\begin{aligned} r &= \log\left(\frac{x+1}{1-x}\right) \\ e^r &= \frac{x+1}{1-x} \\ (1-x)e^r &= x+1 \\ e^r - 1 &= (1+e^r)x \\ x &= \frac{e^r-1}{e^r+1} = \frac{e^{r/2}(e^{r/2}-e^{-r/2})}{e^{r/2}(e^{r/2}+e^{-r/2})} = \frac{\sinh(r/2)}{\cosh(r/2)} = \tanh(r/2), \end{aligned}$$

so the Euclidean radius of $C_{0,r}$ is $x = \tanh(r/2)$.

(c) Any circle is uniquely defined by three points, hence the (hyperbolic) circle has to be equal to the Eucldidean circle with Euclidean center $x/2$ and Euclidean radius $x/2$. Since the hyperbolic distance is invariant under reflection along the real axis, the hyperbolic center point of the circle has to lie on the real axis. Let $p \in (0,1)$ be the hyperbolic center point. We know

$$d_{B_1}(0,p) = d_{B_1}(p,x).$$

We could use the cross ratio formula, or any other that we saw in class, such as the one from Sheet 9, Exercise 1

$$\frac{(p+1)(1-0)}{(0+1)(1-p)} = d_{B_1}(0,p) = d_{B_1}(p,x) = \frac{(x+1)(1-p)}{(p+1)(1-x)}.$$

We solve for $p$

$$\begin{aligned}
(p+1)^2(1-x) &= (x+1)(1-p)^2 \\
xp^2 - 2p + x &= 0 \\
p_{1,2} &= \frac{2 \pm \sqrt{4-4x^2}}{2x} = \frac{1 \pm \sqrt{1-x^2}}{x}
\end{aligned}$$

and we know that $0 < p < 1$, so

$$p = \frac{1-\sqrt{1-x^2}}{x}$$

is the hyperbolic center.

The hyperbolic distance is

$$\begin{aligned}
d_H(0,p) &= \log\frac{1+p}{1-p} = \log\frac{x+1-\sqrt{1-x^2}}{x-1+\sqrt{1-x^2}} \cdot \frac{(x-1)-\sqrt{1-x^2}}{(x-1)-\sqrt{1-x^2}} \\
&= \log\frac{x^2-1+1-x^2+\sqrt{1-x^2}(-x-1-x+1)}{x^2-2x+1-1+x^2} \\
&= \log\frac{\sqrt{1-x^2}}{1-x} = \log\frac{\sqrt{(1-x)(1+x)}}{\sqrt{(1-x)(1-x)}} \\
&= \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log\frac{1+x}{1-x}.
\end{aligned}$$

## Exercise 4

A hyperbolic line is determined by its two points in the boundary $\partial B_1 = S^1$.

(a) Identify the set of hyperbolic lines as quotient of a subset of $S^1 \times S^1$.

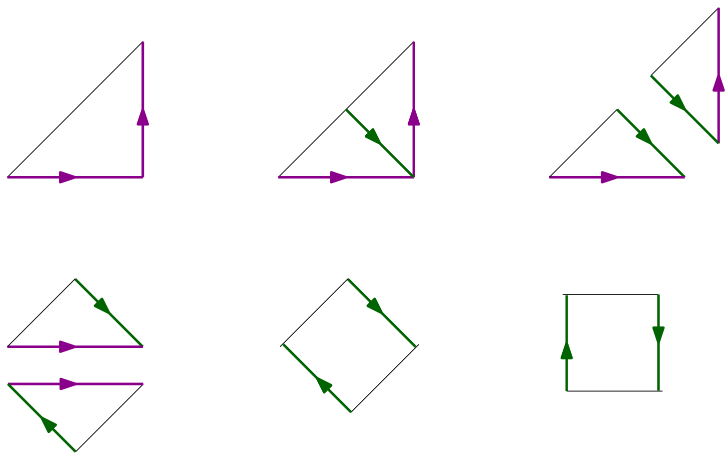
(b) Show that the set of hyperbolic lines (with the topology as a quotient of a subset of $S^1 \times S^1$) is homeomorphic to an open Möbius strip.

**Solution:**

(a) We have a map $\{(x, y) \in S^1 \times S^1 : x \neq y\} \to \{\text{hyperbolic lines}\}$, sending $(x, y)$ to the unique hyperbolic line with endpoints $x$ and $y$. Notice that this map is surjective, but not injective, as $(x, y)$ and $(y, x)$ determine the same hyperbolic line. We thus have a a bijection

$$\{\text{hyperbolic lines}\} \cong \{(x, y) \in S^1 \times S^1 : x \neq y\}/\{(x, y) \sim (y, x)\}.$$

(b) We can draw the torus $S^1 \times S^1$ as a square with opposite sides identified. Since the two endpoints of a geodesic line have to be distinct, we remove the diagonal $\Delta = \{(x, x) \in S^1 \times S^1\}$. The quotient corresponds to glueing the upper-left triangle to the lower bottom triangle by folding along the diagonal. Thus the space of hyperbolic lines can be identified with the triangle $\{(x, y) \colon [0, 1]^2 : x > y\}/\{(x, 0) \sim (1, x)\}$, where the quotient comes from the fact that $(x, 0) \sim (x, 1) \sim (1, x)$. Intuitively we have a rightangled triangle where the hypothenuse is not included and where the two catheta are identified as in the first picture of the following figure.



To see that this is homeomorphic to a Möbius strip we cut along $y = 1 - x$ to get two smaller triangles and now glue the original catheta together. We obtain a square with two opposite sides identified, but in a twist. This is exactly a Möbius strip.