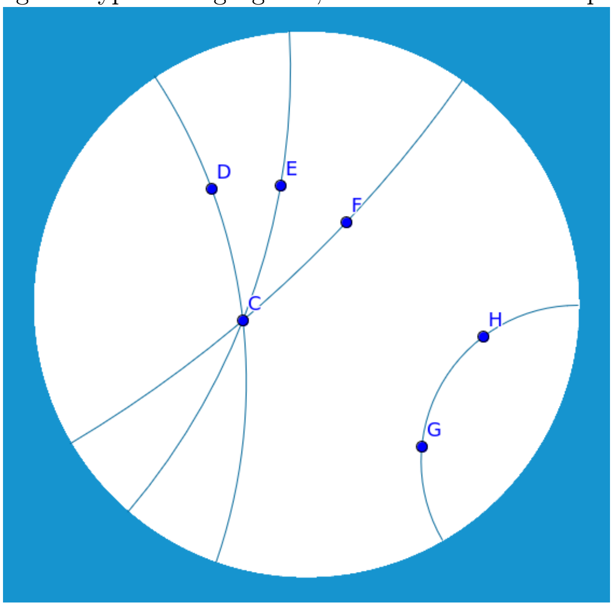
Solutions 12

Exercise 1

Sketch the hypberbolic plane and three pairwise intersecting hyperbolic lines. Sketch a fourth hyperbolic line which is ultraparallel to all previous three.

Solution:
Using the hyperbolic geogebra, we can draw for example:



Exercise 2

Let ℓ and ℓ' be two hyperbolic lines that have a common limit point $p \in \partial B_1 = S^1$. Prove that there are sequences of points $x_1, x_2, \ldots \in \ell$ and $y_1, y_2, \ldots \in \ell'$ with $\lim_{n\to\infty}(x_n) = p = \lim_{n\to\infty}(y_n)$, such that there is a constant C with

$$d_H(x_n, y_n) \leq Ce^{-d_H(x_1,x_n)} \qquad \text{for all } n \in \mathbb{N},$$

i.e. the distance between the hyperbolic lines ℓ and ℓ' converges to 0 exponentially fast.

Hint: Use the Taylor expansion of cosh.

Solution:

Since $\text{Möb}(B_1)$ acts transitively on hyperbolic lines, we may assume that $\ell = (-1, 1) \subseteq B_1$. Up to rotating by 180°, we may also assume that ℓ and ℓ' have the common limit point $p = 1$. Next, by transitivity, we know that ℓ and ℓ' are related by a Möbius transformation and unless $\ell = \ell'$, this Möbius transformation has to be parabolic, as it fixes exactly one point (namely p) in the boundary (compare Sheet 10, exercise 2). We know an explicit parabolic transformation (again from Sheet 10, exercise 2), namely

$$f_b(z) = \frac{(2i-b)z+b}{-bz+(2i+b)}$$

for $b \in \mathbb{R}$. We note that $f_b(p) = p = 1$ and

$$f_b(-1) = \frac{-(2i-b)+b}{b+(2i+b)} = \frac{b-i}{b+i}$$

and claim that for every point $q \in S^1 \setminus \{1\}$, we can find $b \in \mathbb{R}$ such that $f_b(-1) = q$, namely:

$$\begin{aligned} q &= \frac{b-i}{b+i} \\ q(b+i) &= b-i \\ b(q-1) &= -i(1+q) \\ b &= -i\frac{q+1}{q-1} \qquad \text{(we used } q \neq 1\text{)} \end{aligned}$$

If now q is the second endpoint of the geodesic ℓ', we can choose the b as above to get $f_b(-1) = q$. Since hyperbolic lines are determined by their endpoints, we have $f_b(\ell) = \ell'$.

So up to now, we have reduced the situation to the case that $\ell = (-1,1)$, $p = 1$ and $\ell' = f_b(\ell)$ for some $b \in \mathbb{R}$. Now consider $t \in [0,1) \subseteq \ell$ and

$$f_b(t) = \frac{(2i-b)t+b}{-bt+(2i+b)} = \frac{b(1-t)+2ti}{b(1-t)+2i} \in \ell'.$$

We want to calculate

$$d_H(t, f_b(t)) = \operatorname{arccosh}\left(1 + \frac{2|t-f_b(t)|^2}{(1-|t|^2)(1-|f_b(t)|^2)}\right)$$

for which we start with intermediate steps.

$$\begin{aligned} |t-f_b(t)|^2 &= \left|\frac{b(1-t)t+2it-b(1-t)-2ti}{b(1-t)+2i}\right|^2 \\ &= \left|\frac{(t-1)(t+1)b}{b(1-t)+2i}\right|^2 \\ &= \frac{b^2(1-t)^4}{b^2(1-t)^2+4}, \\ 1-|t|^2 &= 1-t^2 \\ 1-|f_b(t)|^2 &= \frac{b^2(1-t)^2+4-b^2(1-t)^2-4t^2}{b^2(1-t)^2+4} = \frac{4(1-t^2)}{b^2(1-t)^2+4}. \end{aligned}$$

Plugging in, we get

$$\begin{aligned} d_H(t, f_b(t)) &= \operatorname{arccosh}\left(1 + \frac{2\frac{b^2(1-t)^4}{b^2(1-t)^2+4}}{(1-t^2)\frac{4(1-t^2)}{b^2(1-t)^2+4}}\right) \\ &= \operatorname{arccosh}\left(1 + \frac{2b^2(1-t)^4}{4(1-t^2)^2}\right) \\ &= \operatorname{arccosh}\left(1 + \frac{b^2(1-t)^2}{2(1+t)^2}\right) =: \operatorname{arccosh}(\tilde{t}). \end{aligned}$$

Let $x_1 = 0$. Then we have

$$d_H(x_1, t) = \log\frac{1+t}{1-t}.$$

We want to find a constant C such that

$$d_H(t, f_b(t)) \leq Ce^{-d_H(x_1,t)} = Ce^{-\log\frac{1+t}{1-t}} = C\left(\frac{1+t}{1-t}\right)^{-1} = C\frac{1-t}{1+t}.$$

Recall the Taylor series

$$\cosh(d) = \frac{e^d + e^{-d}}{2} = 1 + \frac{d^2}{2} + \frac{d^4}{4!} + \dots\ .$$

We see that taking $C = b$ results in

$$\tilde{t} = 1 + \frac{b^2(1-t)^2}{2(1+t)^2} \leq 1 + \frac{1}{2}\left(C\frac{1-t}{1+t}\right)^2 + \frac{1}{4!}\left(C\frac{1-t}{1+t}\right)^4 + \dots = \cosh\left(C\frac{1-t}{1+t}\right)$$

Applying arccosh to both sides results in the required equation

$$d_H(t, f_b(t)) \leq C\frac{1-t}{1+t} = Ce^{-d_H(x_1,t)}.$$

The exercise was formulated in terms of sequences, we can just take a sequence like $x_n = 1 - \frac{1}{n}$ and $y_n = f_b(x_n)$ which satisfy the same exponential convergence.

Exercise 3

(a) Prove that every hyperbolic triangle has angle sum less than 180°.

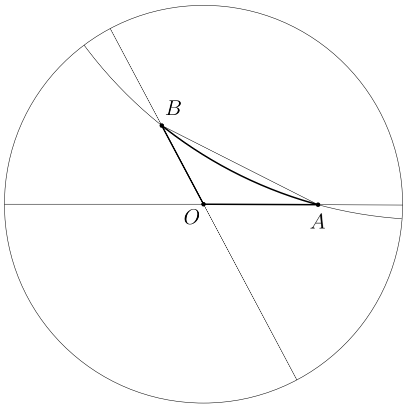
(b) Show that there are hyperbolic triangles of arbitrary small positive interior angle sum.

(c) Prove that there is a regular[1] octagon in the hypberbolic plane, all of whose angles are 45°.

[1] An n-gon is regular if all its sides have the same lengths and the angles at all vertices is the same.

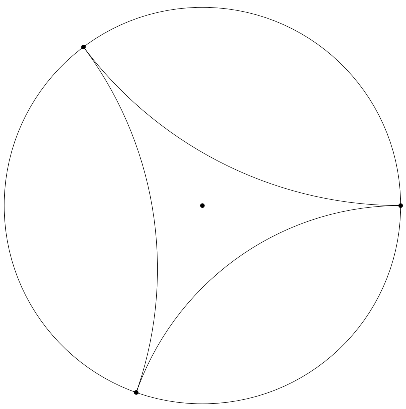
Solution:

(a) Since Möb(B_1) acts transitively on B_1, and preserves angles, we may assume that one of the vertices of the triangle is 0. After applying rotations we may also assume that one of the edges of the triangle lies on the real axis. We know that two of the edges of the triangle (which are segments of hyperbolic lines), are Euclidean straight line segments. The last edge is a segment of a hyperbolic line that is a Euclidean circle segment. The situation looks as follows.



When we place a Euclidean straight line segment between A, B we know that the interior angle sum of the Euclidean triangle OAB is $180°$. Since the Euclidean segment AB is also a secant of the circle defining the hyperbolic line AB, we know that the hyperbolic angles $\angle OAB$ and $\angle OBA$ are smaller than the Euclidean angles. The angle $\angle AOB$ is the same for Euclidean and hyperbolic, so in conclusion we see that the hyperbolic angle sum is less than $180°$.

(b) As we let the three vertices of the triangle go towards the boundary, the angle interior sum tends to 0. The extreme case is that of an ideal triangle, whose vertices are all the way at ∞:



(c) We start with a circle centered at 0 and of Euclidean radius $r \in (0, 1)$.

We can find eight regularly spaced points on this circle (for example by picking a point and then rotating it repeatedly by 45°). These eight points give rise to a regular hyperbolic octagon. We note that the angles are all the same but depend continuously on r. As $r \to 1$, the angle becomes 0. As $r \to 0$, the hyperbolic segments of the octagon approximate Euclidean segments and thus the hyperbolic angle tends to the Eucldiean angle of a octagon, which is 120°. By adjusting r, we can reach any angle between 0° and 120° for the hyperbolic octagon, in particular, there exists r such that the angle is 45°, compare also Figure 1.

Exercise 4

Consider the regular hyperbolic octagon all of whose angles are $2\pi/8$ from Exercise 3(c). Label the sides of the hyperbolic octagon by the letters a, b, a^{-1}, b^{-1}, c, d, c^{-1}, d^{-1} as in Figure 1. Now for each letter in $\{a, b, c, d\}$ glue together the two sides labelled by it and its inverse, respecting the orientation[2]. Denote the resulting object by X.

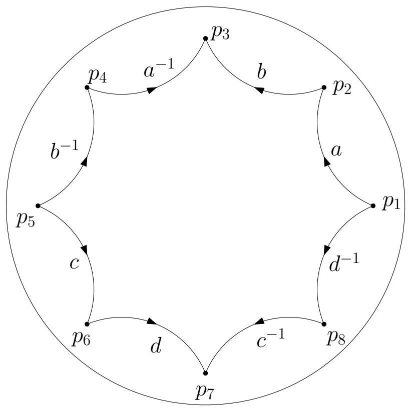
(a) Prove that all eight vertices of the hyperbolic octagon get identified into one point in X.

(b) Show that for every $x \in X$, we can identify a neighborhood of x with a neighborhood of a point in the hyperbolic plane. This way we can give X a local hyperbolic metric, we then say that X is a *hyperbolic surface.*

(c) Show that X is homeomorphic to a double torus.

Solution:

(a) Label the points as follows:



[2]This construction is analogous to the construction of a torus by identifying the opposite sides of a square.

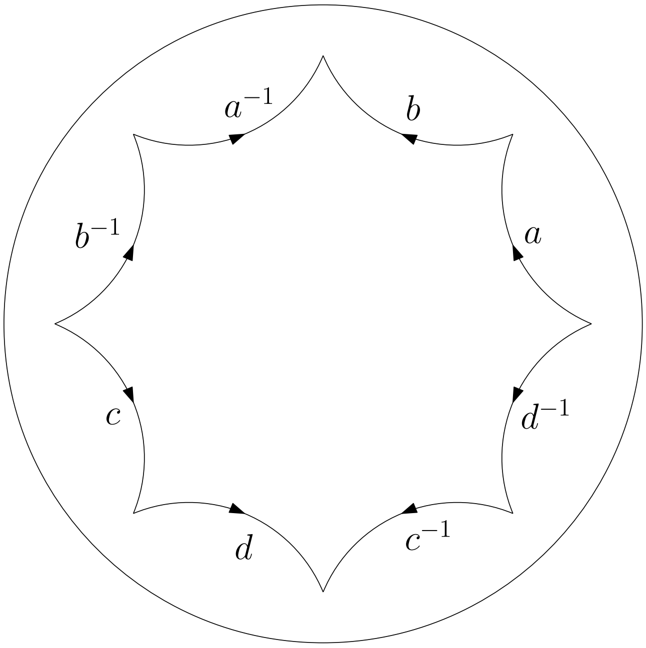


Figure 1: A hyperbolic octagon whose sides are to be identified.

The point p_1 gets identified with p_4, when we glue a. p_4 gets identified with p_3 when we glue b. p_3 gets identifed with p_2 when we glue a, p_2 gets identified with p_5 when we glue b, p_5 gets identified with p_8 when we glue c, p_8 gets identified with p_7 when we glue d, p_7 gets identified with p_6 when we glue c, and finally p_6 gets identified again with p_1 when we glue along d.

(b) For points in the interior of the octagon, they already have a neighborhood which is a neighborhood of the hyperbolic plane. For points on the interior of one of the sides of the octagon, they have half a hyperbolic neighborhood, but in X, when two sides are glued together, these points have a neighborhood which is isometric to a neighborhood of the hyperbolic plane.

Finally due to (a), there is one point in X whose preimage is all the vertices of the octagon. Since the angle in the octagon is $2\pi/8$ and there are 8 pieces to be glued together, this again gives a full nice hyperbolic neighborhood.

(c) Here is a proof by pictures of this exercise: