Solutions 2

Exercise 1

Identify $\mathbb{R}^2 \cong \mathbb{C}$.

(a) Show that the inverse function

$$\begin{aligned} \mathbb{C} \setminus \{0\} &\to \mathbb{C} \setminus \{0\} \\ z &\mapsto 1/z \end{aligned}$$

is an orientation-preserving map.

(b) Define $I \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by extending the inverse to have suitable values at 0 and at ∞. Prove that I is a homeomorphism of $\hat{\mathbb{C}}$ with respect to the topology defined in class.

Solution:

(a) For $z = x + iy$, we have

$$\frac{1}{z} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2},$$

so the map can be reformulated as a real function

$$\begin{aligned} f \colon\ \mathbb{R}^2 \setminus \{0\} &\to \mathbb{R}^2 \setminus \{0\} \\ (x,y) &\mapsto \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (f_1(x,y), f_2(x,y)) \end{aligned}$$

for which we can calculate the derivative at $(a,b) \in \mathbb{R}^2 \setminus \{0\}$ as

$$Df(a,b) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) \\ \frac{\partial f_1}{\partial y}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \end{pmatrix} = \begin{pmatrix} \frac{b^2-a^2}{(a^2+b^2)^2} & \frac{-2ab}{(a^2+b^2)^2} \\ \frac{2ab}{(a^2+b^2)^2} & \frac{b^2-a^2}{(a^2+b^2)^2} \end{pmatrix}$$

with positive determinant

$$\frac{1}{(a^2+b^2)^2}((b^2-a^2)^2 + 4a^2b^2) = \frac{1}{(a^2+b^2)^2}(a^4 + 2a^2b^2 + b^4) = 1$$

which means that f is orientation-preserving.

Alternatively we could use the theorem from the lecture, that holomorphic functions with $f'(z) \neq 0$ are orientation-preserving at z.

(b) We define $I(0) = \infty$ and $I(\infty) = 0$. It is clear that I is a bijection, in fact $I^{-1} = I$. We thus just have to show that I is continuous. The topology on $\hat{\mathbb{C}}$ defined in class is the one-point-compactification, whose open sets consist of the open sets of $\mathbb{C}$ as well as any complement of a closed bounded subset of $\mathbb{C}$ together with the point ∞. We know from analysis, that the map $z \mapsto 1/z$ is continuous on $\mathbb{C} \setminus \{0\}$. To show that

I is continuous everywhere we have to show that preimages of open sets are open.

Fact 1. By continuity of the map $z \mapsto 1/z$ on $\mathbb{C} \setminus \{0\}$, we know that images of open balls $B(z, r)$ (with $r < |z|$) are contained in open balls again. In particular, open sets $U \subseteq \mathbb{C} \setminus \{0\}$ are sent to open sets under I.

Fact 2. An open ball $B(0, r)$ around 0 is sent to $I(B(0, r)) = \{z \in \mathbb{C} \colon |z| > 1/r\} \cup \{0\}$, whose complement is bounded (by $1/r$) and closed (since $I(B(0, r) \setminus \{0\})$ is open). Hence $I(B(0, r))$ is open in the one-point-compactification $\hat{\mathbb{C}}$.

Fact 3. Starting with an open set $U \subseteq \hat{\mathbb{C}}$ containing ∞, we know by definition that its complement C is a closed bounded subset of $\mathbb{C}$. Let $R > 0$ be such that $C \subseteq B(0, R)$. Then $B(0, 1/R)$ is contained in $I(U)$, since $|z| \geq R$, if and only if $|I(z)| < 1\mathbb{R}$. Since C is closed, $\mathbb{C} \setminus C = U \setminus \{\infty\}$ is open. Thus $I(U \setminus \{\infty\})$ is open and so is $I(U \setminus \{\infty\}) \cup I(\{\infty\}) = I(U)$.

By these three facts, we can conclude that the image of every open subset $U \subseteq \hat{\mathbb{C}}$ is open, and hence I is continuous.

Exercise 2

Consider the three Möbius transformations $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$I \colon z \mapsto \frac{1}{z} \qquad J \colon z \mapsto \frac{1}{\bar{z}} \qquad C \colon z \mapsto \bar{z}.$$

(a) Describe the group generated by I, J and C.

(b) Describe the actions of I, J and C on the Riemann sphere considered as the round sphere S^2.

(c) Which of these maps are orientation-preserving? Which are orientation-reversing?

Solution:

(a) We have $\mathrm{id} = I^2 = J^2 = C^2$ adn $J \circ I = C, I \circ C = J$ and $C \circ J = I$, so the group generated by I, J, C consist of the four elements id, I, J, C. Its group table is given by

$\circ$	id	I	J	C
id	id	I	J	C
I	I	id	C	J
J	J	C	id	I
C	C	J	I	id

and as an abstract group it is isomorphic to the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(b) The stereographic projection can be used to translate between the sphere S^2 and $\hat{\mathbb{C}}$. The maps I and J flip the interior of the unit circle with the outside, this corresponds to flipping the upper hemisphere of S^2 with the lower hemisphere. For a point $z = x + iy$ on the unit circle, we have

$$I(z) = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{|z|^2} = \bar{z},$$

so it flips the imaginary axis. On the sphere S^2 the action of I corresponds to reflecting along the equator, and then also reflecting on the great-circle corresponding to the real axis on $\hat{\mathbb{C}}$. Compare figure 1.

For a point $z = x + iy$ on the unit circle, we also have

$$J(z) = \frac{1}{x-iy}\frac{x+iy}{x+iy} = \frac{x+iy}{|z|^2} = z,$$

which means that J fixes the point on the unit circle, i.e. the action of J on S^2 is just the reflection on the equator.

Finally, C flips $\mathbb{C}$ along the real axis, this correponds to a reflection an the great circle corresponding to the real axis in S^2.

(c) I is an orientation-preserving Möbius transformation, while J and C are orientation-reversing.

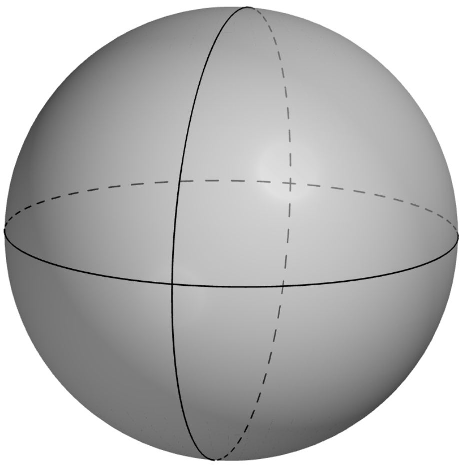

Figure 1: The sphere S^2 with the equator and a great circle corresponding to the real axis together with ∞ in $\hat{\mathbb{C}}$.

Exercise 3

(a) Give an example of a real affine map $\mathbb{R}^2 \to \mathbb{R}^2$ that is not a similarity.

(b) Classify the similarities of $\mathbb{R}^2$ in terms of their fixed points.

(c) Show that the group of similarities of the plane $\mathrm{Sim}(\mathbb{R}^2)$ has the structure of a semidirect product $\mathrm{Sim}(\mathbb{R}^2) = \mathrm{Sim}_+(\mathbb{R}^2) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Solution:

(b) The map $(x, y) \mapsto (2x, y)$ is a real affine map (it is the multiplication with the diagonal matrix $\mathrm{Diag}(2, 1)$ and translation by vector $v = 0$), but it is not a similarity, as the distance $d(0, (1, 0))$ is scaled by a factor of 2, while the distance $d(0, (0, 1))$ is only scaled by a factor of 1.

One can show that a real affine map $x \mapsto Ax+v$ is a similarity exactly when A is an orthogonal matrix.

(c) There are four cases.

1. If a similarity fixes all points, it is the identity.

2. If it fixes at least two points, then the scaling factor has to be 1, i.e. the similarity is an isometry. By geometric considerations it then also has to fix the whole line spanned by the two points. An isometry fixing a whole line pointwise has to be a reflection along that line (or the identity).

3. If a similarity fixes exactly one point, it could be a rotation (or reflection) followed by any dilation.

4. A similarity could also fix no points, it could then be a translation or a mix of rotations, dilations and translations.

(c) We note that $\mathrm{Sim}_+(\mathbb{R}^2)$ is a normal subgroup of $\mathrm{Sim}(\mathbb{R}^2)$, since conjugation sends an orientation-preserving element to an orientation-preserving element. We now take an orientation reversing similarity, such as $s\colon z \mapsto \bar{z}$ and view the group generated by s as $\mathbb{Z}/2\mathbb{Z}$, (since $s^2 = \mathrm{id}$). We first have to show that every similarity is a combination of an orientation-preserving similarity and either the identity or s. Indeed, this is the case, since similarities are either orientation-preserving (in which case we are done) or orientation-reversing, in which case, we can precompose s to obtain an orientation preserving similarity. In this sense we have as sets

$$\begin{aligned} \mathrm{Sim}_+(\mathbb{R}^2) \times \mathbb{Z}/2\mathbb{Z} &\equiv \mathrm{Sim}(\mathbb{R}^2) \\ (\varphi, \mathrm{id}) &\mapsto \varphi \\ (\varphi, t) &\mapsto \varphi \circ s \end{aligned}$$

However $\mathrm{Sim}(\mathbb{R}^2)$ viewed as a group is not the direct product, since for example $(z \mapsto z - i, \mathrm{id}) = (\mathrm{id}, s) \circ (z \mapsto z + i, s) \neq (z \mapsto z + i, s^2)$.

To give this set the structure of a semidirect product, we have to define an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Sim}_+(\mathbb{R}^2)$, which we choose to be the conjugation: $x.\varphi := x \circ \varphi \circ x^{-1} = x \circ \varphi \circ x$ for $x \in \{\mathrm{id}, s\} = \mathbb{Z}/2\mathbb{Z}$ and $\varphi \in \mathrm{Sim}_+(\mathbb{R}^2)$. With this action, the multiplication in the semidirect product is defined as

$$\begin{aligned} \circ_{\rtimes} : (\mathrm{Sim}_+(\mathbb{R}^2) \rtimes \mathbb{Z}/2\mathbb{Z}) \times (\mathrm{Sim}_+(\mathbb{R}^2) \rtimes \mathbb{Z}/2\mathbb{Z}) &\to \mathrm{Sim}_+(\mathbb{R}^2) \rtimes \mathbb{Z}/2\mathbb{Z} \\ ((\varphi, x), (\psi, y)) &\mapsto (\varphi \circ x.\psi, x \circ y) \end{aligned}$$

for $x, y \in \mathbb{Z}/2\mathbb{Z}$ and $\varphi, \psi \in \mathrm{Sim}_+(\mathbb{R}^2)$. We verify that this is the correct group operation corresponding to the group operation in $\mathrm{Sim}(\mathbb{R}^2)$: If $(\varphi, x) \equiv \varphi \circ x$ and $(\psi, y) \equiv \psi \circ y$ are elements of $\mathrm{Sim}(\mathbb{R}^2)$, we have

$$\begin{aligned} (\varphi, x) \circ_{\rtimes} (\psi, y) &= (\varphi \circ x.\psi, x \circ y) = (\varphi \circ x \circ \psi \circ x^{-1}, x \circ y) \\ &\equiv \varphi \circ x \circ \psi \circ x^{-1} \circ x \circ y \\ &= (\varphi \circ x) \circ (\psi \circ y) \end{aligned}$$

which is the correct group operation. Thus we have shown that $\mathrm{Sim}(\mathbb{R}^2) = \mathrm{Sim}_+(\mathbb{R}^2) \rtimes \mathbb{Z}/2\mathbb{Z}$.