Solutions 3

Exercise 1

(a) Show that orientation-preserving Möbius transformations $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ have exactly either one or two fixed points, or are the identity.

(b) Give examples of orientation-preserving Möbiustransformations with one (resp. two) fixed points.

(c) Draw a picture of how the Möbiustransformations from (b) act on $\hat{\mathbb{C}}$.

(d) Does the statement of (a) also hold for orientation-reversing Möbius transformations?

Solution:

(a) Let f be a orientation-preserving Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$.

We are first careful with ∞. We note that ∞ can only be a fixed point of f, when $c = 0$. In this case, any other fixed point satisfies

$$\frac{a}{d}z + \frac{b}{d} = z,$$

which means that there is either NO second fixed point (if $a/d = 1$ and $b/d \neq 0$), or ALL points are fixed (if $a/d = 1$ and $b/d = 0$), or (if $a/d \neq 1$) at most ONE second fixed point, namely $z = b/(d-a)$.

If ∞ is no fixed point, ($c \neq 0$), then any fixed point is described by the equality

$$\frac{az+b}{cz+d} = z$$

and since $z \neq \infty$, we know that $cz + d \neq 0$, hence

$$\begin{aligned} az + b &= cz^2 + dz \\ 0 &= cz^2 + (d-a)z - b \end{aligned}$$

which is a quadratic equation and thus has exactly one or two solutions over $\mathbb{C}$.

(b) Examples of Möbius transformations with one fixed point are $z \mapsto z + 1$, where the fixed point is ∞, or $z \mapsto z/(z+1)$, where the fixed point is 0.

Examples with two fixed points could be $z \mapsto 2z$, where the fixed points are ∞ and 0, or $z \mapsto 1/z$ with fixed points ± 1.

(c) See figures 2 and 1.

(d) No, statement (a) may not hold, as orientation reversing maps may aso fix circles or lines, for example $z \mapsto \bar{z}$ fixes the real axis.

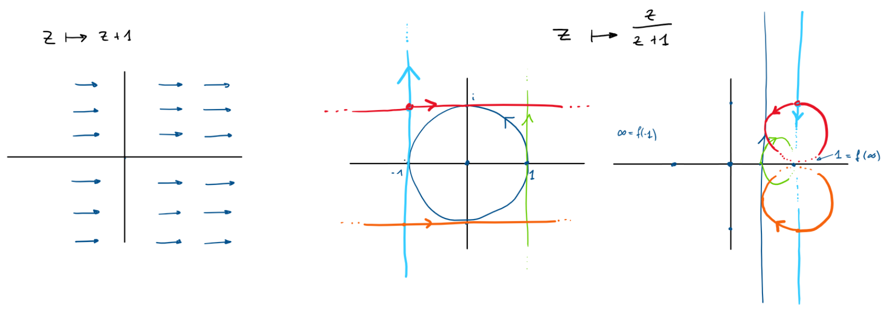

Figure 1: On the left, the map $z \mapsto z + 1$ with fixed point ∞ is illustrated. On the right, the function $z \mapsto z/(1 + z)$ is illustrated by means of showing the domain on the left and the target on the right. The fixed point is 0.

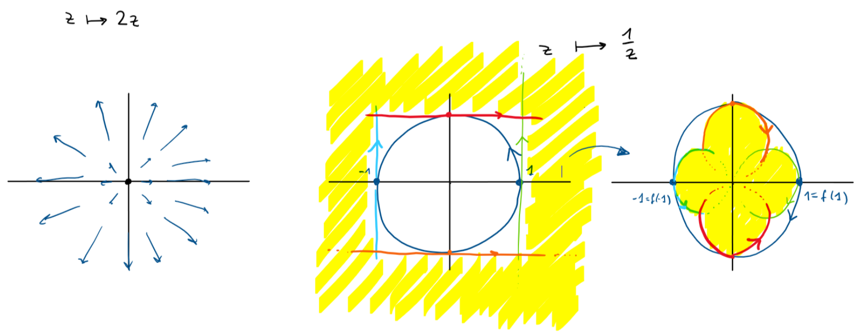

Figure 2: On the left, the map $z \mapsto 2z$ with fixed points ∞ and 0 is illustrated. On the right, the function $z \mapsto 1/z$ is illustrated by means of showing the domain on the left and the target on the right. The fixed points are $1, -1$.

Exercise 2

(a) Show that the group of Möbius transformations Möb acts transitively on $\hat{\mathbb{C}}$, i.e. for any $x, y \in \hat{\mathbb{C}}$ there is a $g \in$ Möb such that $g(x) = y$.

(b) Determine all the combinations of coefficients a, b, c, d such that the corresponding Möbiustransformation is the identity.

(c) Find all orientation-preserving Mobius transformations $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that send
$$\infty \mapsto i \mapsto 0 \mapsto -i \mapsto \infty.$$

(d) What are the possible values for $f(1)$ under a Mobius transformation as in (c)?

Solution:

(a) We first show that there is a Möbius transformation sending $0 \in \hat{\mathbb{C}}$ to any point $p \in \hat{\mathbb{C}}$. If $p \in \mathbb{C}$, then we take the Möbiustransformation
$$z \mapsto \frac{1z + p}{0z + 1},$$
if $p = \infty$ we can take
$$z \mapsto \frac{0z + 1}{1z + 0}.$$
If we now want to send p to q, we first take the inverse of the matrix above to send p to 0 and we then use a matrix as above again to take 0 to q. Since groups of maps on sets always act on the sets, we can compose these two group elements to get one that sends p to q.

(b) We have
$$z = \frac{az + b}{cz + d}$$
for all $z \in \hat{\mathbb{C}}$ and plug in a few points:

For $z = 0$, we obtain $b = 0$. For $z = \infty$ we obtain $a/c = \infty$, i.e. $c = 0$. Using these two facts we get that $z = az/d$, hence we also need $a/d = 1$. In fact all Möbius transformations with $a, b = 0, c = 0, d$ that satisfy $a = d \neq 0$ fix all points of $\hat{\mathbb{C}}$ (the condition $ad - bc \neq 0$ is satisfied automatically).

It may be a good exercise to try to translate everything in terms of matrices.

(c) Let $a, b, c, d \in \mathbb{C}$ such that
$$f\colon z \mapsto \frac{az + b}{cz + d}$$
is a orientation-preserving Möbiustransformation that sends $\infty \mapsto i \mapsto 0 \mapsto -i \mapsto \infty$. We plug in ∞ to get $a/c = i$. We plug in i to get $ai + b = 0$. We plug in 0 to get $b/d = -i$ and finally we plug in $-i$ to get $d = ic$. From the first and fourth result we get $a = d$. From $a = ci$ and $b = -ai$ we get $b = c$. Therefore f must be of the form
$$z \mapsto \frac{az - ai}{-aiz + a}.$$
We have $ad - bc = a^2 - (-ai)^2 = 2a^2$, hence this is a Möbius transformation for all $a \neq 0$. For any $a \neq 0$ this function sends $\infty \mapsto i \mapsto 0 \mapsto -i \mapsto \infty$.

(d) We plug in
$$f(1) = \frac{a1 - ai}{-ai1 + a} = 1.$$
This means that any Möbiustransformation that sends $\infty \mapsto i \mapsto 0 \mapsto -i \mapsto \infty$ fixes the point 1.

Exercise 3

Answer the questions (a) - (e) for the functions (**1**) - (**7**). All functions are continuous functions of the Riemann-sphere $\hat{\mathbb{C}}$. For (**1**) - (**5**) we have $f(\infty) = \infty$.

(a) Find the fixed point set $\{z \in \hat{\mathbb{C}} \colon f_i(z) = z\}$.

(b) Does f_i preserve the unit disk?

(c) Is f_i an affine map, when restricted to $\mathbb{C} \cong \mathbb{R}^2$?

(d) Is f_i a similarity, when restricted to $\mathbb{C}$?

(e) If f_i orientation-preserving or orientation-reversing?

(**1**) $f_1 \colon z \mapsto z + a$, where $a \in \mathbb{C}$ fixed.

(**2**) $f_2 \colon z \mapsto rz$, where $r > 0$ fixed.

(**3**) $f_3 \colon z \mapsto e^{i\varphi} z$, where $\varphi \in [0, 2\pi)$ fixed.

(**4**) $f_4 \colon z \mapsto \bar{z}$.

(**5**) $f_5 \colon z \mapsto z^2$.

(**6**) $f_6 \colon z \mapsto \frac{z}{|z|^2}$, where $f(\infty) = 0$ and $f(0) = \infty$.

(**7**) $f_7 \colon z \mapsto \frac{1}{z}$, where $f(\infty) = 0$ and $f(0) = \infty$.

Solution:
Let $F(f)$ be the fixed point set. We use the the notation $z = r \cdot e^{i\varphi} = x + iy$ for the complex numbers.

(1) $F(f_1) = \{\infty\}$, or $F(f_1) = \hat{\mathbb{C}}$, when $a = 0$. The unit disk is not preserved, except when $a = 0$. Yes, $f_1(z) = \mathrm{Id}\, z + a$ and hence is a affine map. Translations are isometries, hence similarities. Yes, translations are orientation-preserving.

(2) $F(f_2) = \{0, \infty\}$, or $F(f_2) = \hat{\mathbb{C}}$, when $r = 1$. The unit disk is not preserved, except when $r = 1$. Yes, $f_2(z) = \mathrm{Diag}(r, r)z + 0$ is an affine map. Yes, scalings are similarities and yes, scalings preserve orientation.

(3) This is a rotation around 0 by the angle φ. We have $F(f_3) = \{0, \infty\}$, or $F(f_3) = \hat{\mathbb{C}}$, when $\varphi = 0$. The unit disk is preserved. Yes, it is an affine map:

$$f_3(z) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} z + 0.$$

Yes, rotations are isometries, hence similarities and yes, rotations preserve orientation.

(4) This is a reflection along the real axis. We have $F(f_4) = \{z \in \mathbb{C} \colon \operatorname{Im}(z) = 0\} \cup \{\infty\}$, the real axis. The unit disk is preserved. It is an affine transformation:

$$f_3(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + 0.$$

Yes, reflections are isometries, hence similarities. Reflections are orientation-reversing.

(5) We calculate $f_5(re^{i\varphi}) = r^2 e^{i2\varphi}$. For z to be a fixed point, we need $r = 1, 0$ or ∞ and $\varphi = 2\varphi$. These two conditions are satisfied for $F(f_5) = \{0, 1, \infty\}$. Yes, the unit disk is preserved, but is sent to itself twice.

No, this function cannot be represented as an affine map. As $f_5(0) = 0$, it would need to be a linear map, if it was affine. Linear maps are determined by the image of a base such as $\{1, i\}$, so since $f_5(1) = 1$ and $f_5(i) = -1$, linearity would then give that $f_5(1 + i) = 0$, but this is not so.

The function is not a similarity, this can be seen for $r > 0$, as $d(0, r) = r \cdot d(f(0), f(r))$, hence there is no constant scaling factor.

The map is orientation preserving everywhere except at 0 and ∞, where it is not smooth.

(6) This is the inversion on the circle. We have $f_6(z) = z$ if and only if $|z|^2 = 1$, so $|z| = 1$, so $F(f_6) = \{z \in \mathbb{C} \colon |z| = 1\}$ is the unit circle. The interior of the unit circle is sent to the exterior, hence the unit disk is not preserved.

The map is not an affine map or a similarity (its restriction on $\mathbb{C}$ is not even well defined). The map f_6 is orientation-reversing, as it is a reflection along a circle.

(7) This function is similar to (6). We can describe it as an inversion on the circle followed by a reflection $z \mapsto \overline{z}$: $f_7 = f_4 \circ f_6$. We see that $f_7(z) = z$ if and only if $z^2 = 1$. This holds for $z = 1$ and $z = -1$. Therefore $F(f_7) = \{-1, 1\}$. Similar to f_6, the unit circle is preserved, but not the unit disk. f_7 cannot be an affine map nor a similarity. As f_7 is the composition of two orientation-reversing maps, it is orientation preserving.