Solution 4

Exercise 1

For the following exercises, first reason geometrically and then find an algebraic description.

(a) Let $p \in \mathbb{C}$. Describe the point-reflection $Q_p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

(b) Show that $Q_p^2 = \mathrm{id}$.

(c) For $p, q \in \mathbb{C}$, what is $Q_p \circ Q_q$?

(d) Let $L \subseteq \mathbb{C}$ be a line. Describe the reflection $R_L : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ along L.

(e) Show that for every line L through 0, R_L can be written as $z \mapsto e^{i\varphi}\bar{z}$ for some $\varphi \in \mathbb{R}$.

(f) What is the line along which $z \mapsto e^{i\varphi}\bar{z}$ reflects?

(g) Write Q_p and R_L as Möbius transformations.

Solution:

(a) Geometrically, the point reflection of z on the point p is defined by drawing a line through z and p, and then taking the unique other point z' on that line that satisfies $d(p, z) = d(p, z')$. One can obtain this, by considering the vector $\vec{pz}$. Then $z = p + \vec{pz}$ and $Q_p(z) = p - \vec{pz}$. Additionally, ∞ should be sent to ∞.

Algebraically, Q_p can be written as

$$Q_p(z) = p - \vec{pz} = p - (z - p) = 2p - z.$$

(b) Geometrically, this is clear. Algebraically we have

$$Q_p(Q_p(z)) = 2p - Q_p(z) = 2p - (2p - z) = z.$$

(c) Geometrically, we can draw a line from p to q. Since both Q_p and Q_q send points on one side of the line to points on the other side, the composition $Q_p \circ Q_q$ preserves the sides of the line. Additionally both Q_p and Q_q preserve the distance of points to the line. So it is now only important to know how points on the line are affected by $Q_p \circ Q_q$, as all other points behave the same, just in parallel. For points z on the line, we notice that $Q_p \circ Q_q(z)$ is translated by $2\vec{qp}$. So $Q_p \circ Q_q = T_{2\vec{qp}}$.

Algebraically this is easier to calculate:

$$Q_p \circ Q_q(z) = Q_p(2q - z) = 2p - (2q - z) = z + 2(p - q) = z + 2\vec{qp}.$$

(d) Geometrically, to reflect a point z along a line L, one has to find a line L_z perpendicular to L that contains z. Then $R_L(z)$ is the other unique point on L_z which has the same distance from the intersection point $L \cap L_z$ as z. As before, $\infty \mapsto \infty$.

Algebraically, if the line L is given by the equation $ax+by+c=0$ for some parameters a, b, c, then the normal vector is $\vec{v} = (a, b)$. Thus for $z = z_1 + iz_2$ the line L_z is parametrized by $z + t\vec{v}$. The intersection point of the two lines satisfies the equation

$$a(z_1 + ta) + b(z_2 + ta) + c = 0,$$

which is satisfied for

$$t_1 = -\frac{z_1 a + z_2 b + c}{a^2 + b^2}.$$

As the distance from z to the $R_L(z)$ is twice as large as the distance from z to the intersection point, we have

$$R_L(z) = z + 2t_1\vec{v} = z - 2\frac{z_1 a + z_2 b + c}{a^2 + b^2}(a + ib)$$

(e) We first note that $z \mapsto \bar{z}$ is the reflection on the real axis.

Given a line L that goes through 0, i.e. $c = 0$, and we may assume without loss of generality that $a^2 + b^2 = 1$. We have

$$\begin{aligned} R_L(z) &= z_1 - 2a(z_1 a + z_2 b) + i\,(z_2 - 2b(z_1 a + z_2 b)) \\ &= z_1(1 - 2a^2) - z_2(2ab) + i\left(z_2(1 - 2b^2) - z_1(2ab)\right) \\ &= (1 - 2a^2 - 2iab)(z_1 - iz_2) = e^{\varphi}\bar{z} \end{aligned}$$

where we used $a^2 + b^2 = 1$ to see that $1 - 2a^2 = 2b^2 - 1$. Finally this number $1 - 2a^2 - 2iab$ is just a complex number and hence can be written as $e^{i\varphi}$ (where $\tan(\varphi) = 1 - 2a^2 - 2iab$).

(f) We want to find out which points are fixed by $z \mapsto e^{i\varphi}\bar{z}$. We have

$$\begin{aligned} z_1 + iz_2 = z = e^{i\varphi}\bar{z} &= (\cos(\varphi) + i\sin(\varphi))(z_1 - iz_2) \\ &= z_1\cos(\varphi) + z_2\sin(\varphi) + i(z_1\sin(\varphi) - z_2\cos(\varphi)) \end{aligned}$$

Comparing the real parts $z_1 = z_1\cos(\varphi) + z_2\sin(\varphi)$, we get

$$\frac{z_2}{z_1} = \frac{1 - \cos(\varphi)}{\sin(\varphi)}$$

(assuming $\sin(\varphi) \neq 0$), which is the slope of the line

$$y = \frac{1 - \cos(\varphi)}{\sin(\varphi)}x$$

along which $z \mapsto e^{i\varphi}\bar{z}$ reflects.

If $\sin(\varphi) = 0$, then $\cos(\varphi) \in \{\pm 1\}$. If $\cos(\varphi) = -1$, we obtain from comparing the real parts that $z_1 = -z_1$, i.e. $z_1 = 0$, so the line is the vertical line. If $\cos(\varphi) = 1$, then looking at the real part does not help, but comparing the imaginary part we obtain $z_2 = -z_2$, hence $z_2 = 0$, so in this case the line is the real axis.

(g) For the point reflection we get directly

$$Q_p(z) = \frac{-1z + 2p}{0z + 1} = \frac{-z + 2p}{.}$$

For the line reflection, we would like to use $z \mapsto e^{i\varphi}\bar{z}$, but need to take into account that lines may not pass through 0. If L is a line defined by $ax + by + c = 0$, then L_0 defined by $ax + by = 0$ is the parallel line through 0. For L_0 we know by (e) that there is a $\varphi > 0$ such that $R_{L_0}(z) = e^{i\varphi}\bar{z}$. We also know from (d) that

$$\begin{aligned} R_L(z) &= z - 2\frac{az_1 + bz_2 + c}{a^2 + b^2}(a + ib) \\ &= z - 2\frac{az_1 + bz_2}{a^2 + b^2}(a + ib) - 2\frac{c}{a^2 + b^2}(a + ib) \\ &= R_{L_0}(z) - 2\frac{c}{a^2 + b^2}(a + ib) \\ &= e^{i\varphi}\bar{z} - 2\frac{c}{a^2 + b^2}(a + ib) \\ &= \frac{e^{i\varphi}\bar{z} + \left(-2c\frac{(a+ib)}{a^2+b^2}\right)}{0\bar{z} + 1} \end{aligned}$$

which writes R_L as an orientation-reversing Möbius transformation.

Exercise 2

Let $b \in \mathbb{C}$ and $\lambda > 0$. Let T_b be the translation by b and let M_λ be the multiplication by λ.

(a) Describe the effect of $T_b \circ M_\lambda \circ T_{-b}$ geometrically.

(b) Describe the effect of $M_\lambda \circ T_b \circ M_{\lambda^{-1}}$ geometrically.

(c) Express the transformations in (a) and (b) as Möbius transformations.

Solution:

(a) While M_λ is a dilation by a factor λ centered at 0, $T_b \circ M_\lambda \circ T_{-b}$ is a dilation by a factor λ centered at b. The conjugation by T_b just changes the center of the dilation but does not change the *type* of transformation, (note $T_{-b} = (T_b)^{-1}$).

Symbolically, one can define $f\colon \mathbb{C} \to \mathbb{C}$ to be a dilation by a factor λ centered at p if for all $q \in \mathbb{C}$: $d(f(p), f(q)) = \lambda d(p,q)$ and f preserves lines trough p. It is clear that M_λ is a dilation. We claim that $f = T_b \circ M_\lambda \circ T_{-b}$ is a dilation by a factor λ centered at b. We first note that $f(b) = b$.

$$\begin{aligned} f(q) &= b + \lambda(q-b) \\ d(f(b), f(q)) &= |f(q) - f(b)| = |b + \lambda(q-b) - b| = \lambda|q-b| = \lambda d(b,q). \end{aligned}$$

Lines through b are preserved under f, since for all $t \in \mathbb{R}$ and $v \in \mathbb{C}$,

$$f(b+tv) = b + \lambda((b+tv) - b) = b + (\lambda t)v$$

describes the same line.

(b) While T_b is a translation by the element $b \in \mathbb{C}$, $M_\lambda \circ T_b \circ M_{\lambda^{-1}}$ is also a translation, but by the element λb. The conjugation by M_λ just changes by how much the translation acts, but it does not change the *type* of transformation, (note $M_{\lambda^{-1}} = (M_\lambda)^{-1}$).

Symbolically, we have

$$M_\lambda \circ T_b \circ M_{\lambda^{-1}}(z) = \lambda(b + \lambda^{-1}z) = \lambda b + z = T_{\lambda b}(z).$$

(c) We have

$$\begin{aligned} T_b \circ M_\lambda \circ T_{-b}(z) &= \frac{\lambda z + (1-\lambda)b}{0z+1} \\ M_\lambda \circ T_b \circ M_{\lambda^{-1}} = T_{\lambda b} &= \frac{z + \lambda b}{0z+1}. \end{aligned}$$

Exercise 3

Given a group of matrices $\mathrm{G} \subseteq M^{n\times n}(\mathbb{C})$, let $Z_{\mathrm{G}} = \{g \in G\colon \forall h \in G\colon gh = hg\}$ be the center. Then PG is defined to be the group $\mathrm{G}/Z_{\mathrm{G}}$. Recall that for any field F and any $n \geq 1$,

$$\begin{aligned} \mathrm{GL}(n,F) &= \{g \in M^{n\times n}(F)\colon \det(g) \neq 0\} \\ \mathrm{SL}(n,F) &= \{g \in M^{n\times n}(F)\colon \det(g) = 1\}. \end{aligned}$$

(a) Show that $\mathrm{PGL}(2,\mathbb{R}) \ncong \mathrm{PSL}(2,\mathbb{R})$.

(b) Show that $\mathrm{PGL}(3,\mathbb{R}) \cong \mathrm{PSL}(3,\mathbb{R})$.

(c) What about $\mathrm{PGL}(n,\mathbb{R})$ and $\mathrm{PSL}(n,\mathbb{R})$ for $n \geq 4$?

(d) Show that $\mathrm{PGL}(n,\mathbb{C}) \cong \mathrm{PSL}(n,\mathbb{C})$ for all $n \geq 2$.

Solution:
We need to calculate $Z_{\mathrm{GL}(n,\mathbb{F})}$ and $Z_{\mathrm{SL}(n,\mathbb{F})}$ for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. We know that the matrix $\lambda \,\mathrm{Id}_n$ for $\lambda \in \mathbb{F}$ commutes with every other matrix of the same size. We claim that $A_\lambda = \lambda \,\mathrm{Id}_n$ are the only matrices that commute with every other matrix B. Let $A \in Z_{\mathrm{G}}$. For $i \neq j$, let B_{ij} be the matrix that has ones on the diagonal and a 1 in the entry (i, j) and 0 everywhere else (note that B_{ij} is contained in all the groups that we consider). We consider $B_{ij}A = AB_{ij}$, and in particular we consider the entry (i, j):

$$(B_{ij}A)_{ij} = \sum_k b_{ik}a_{kj} = b_{ii}a_{ij} + b_{ij}a_{jj} = a_{ij} + a_{jj}$$
$$(AB_{ij})_{ij} = \sum_k a_{ik}b_{kj} = a_{ij}b_{jj} + a_{ii}b_{ij} = a_{ij} + a_{ii}$$

which implies that $a_{ii} = a_{jj}$.

Next we consider the diagonal matrices $B = \mathrm{Diag}(e^{2n}, e^{2n-2}, \dots, e^{-2n})$ (if n even) or $B = \mathrm{Diag}(e^{2n+1}, e^{2n-1}, \dots, e^{-2n-1})$, (if n odd). Note that B is contained in all the groups we consider. We consider again the entry (i, j) (for $i \neq j$) of the matrix $BA = AB$ to get

$$(AB)_{ij} = \sum_k a_{ik}b_{kj} = a_{ij}b_{jj}$$
$$(BA)_{ij} = \sum_k b_{ik}a_{kj} = b_{ii}a_{ij},$$

which are equal only if $a_{ij} = 0$.

The two considerations above show that any $A \in Z_{\mathrm{G}}$ has to be of the form $\lambda \,\mathrm{Id}_n$ for $\lambda \in \mathbb{F}$. Hence

$$Z_{\mathrm{G}} = \{A \in \mathrm{G} \colon A = \lambda \,\mathrm{Id}_n \text{ for } \lambda \in \mathbb{F}\}$$

(a) We have

$$Z_{\mathrm{GL}(2,\mathbb{R})} = \{\lambda \,\mathrm{Id}_2 \colon \lambda \in \mathbb{R} \setminus \{0\}\}$$
$$Z_{\mathrm{SL}(2,\mathbb{R})} = \{\lambda \,\mathrm{Id}_2 \colon \lambda = 1 \text{ or } \lambda = -1\}$$

We consider the map φ defined by first taking the inclusion and then the projection.

$$\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{GL}(2,\mathbb{R}) \twoheadrightarrow \mathrm{PGL}(2,\mathbb{R})$$
$$\varphi \colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R})$$

We first show $\mathrm{Ker}(\varphi) = Z_{\mathrm{SL}(2,\mathbb{R})}$: Let $g \in \mathrm{SL}(2,\mathbb{R})$ such that $\varphi(g) = [\mathrm{Id}_2]$. This means that $g \in Z_{\mathrm{GL}(2,\mathbb{R})}$, i.e. $g = \lambda \,\mathrm{Id}_2$. Since $g \in \mathrm{SL}(2,\mathbb{R})$, $\lambda \in \{\pm 1\}$, i.e. $g \in Z_{\mathrm{SL}(2,\mathbb{R})}$. On the other hand, if $g \in Z_{\mathrm{SL}(2,\mathbb{R})}$, then $g = \lambda \,\mathrm{Id}_2$ with $\lambda \in \{\pm 1\}$ and thus $g \in Z_{\mathrm{GL}(2,\mathbb{R})}$, hence $g \in \ker(\varphi)$. By the isomorphism theorem we have

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\ker(\varphi) \cong \mathrm{im}(\varphi) \subseteq \mathrm{PGL}(2,\mathbb{R}).$$

To show that $\mathrm{PSL}(2,\mathbb{R}) \neq \mathrm{PGL}(2,\mathbb{R})$, we thus just have to show that φ is not surjective: Consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \setminus \mathrm{SL}(2,\mathbb{R})$$

For any $\lambda \,\mathrm{Id}_2 \in Z_{\mathrm{GL}(2,\mathbb{R})}$, we have

$$\det(A\lambda \,\mathrm{Id}_2) = \det(A)\det(\lambda \,\mathrm{Id}_2) = -1\lambda^2$$

which is always negative, hence $A\lambda \,\mathrm{Id}_2$ is never in $\mathrm{SL}(2,\mathbb{R})$, hence there is no preimage of $[A]$ in $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{R}) \ncong \mathrm{PSL}(2,\mathbb{R})$.

(b) We argue as in (a), it follows analogously that $\ker(\varphi) = Z_{\mathrm{SL}(3,\mathbb{R})}$. We again use the isomorphism theorem

$$\mathrm{PSL}(3,\mathbb{R}) = \mathrm{SL}(3,\mathbb{R})/\ker(\varphi) \cong \mathrm{im}(\varphi) \subseteq \mathrm{PGL}(3,\mathbb{R}).$$

and this time show that φ is surjective. Given any $g \in \mathrm{GL}(3,\mathbb{R})$, consider the third root $\lambda_3 := (\det(g))^{1/3}$ of $\det(g)$. We can define $\lambda_3^{-1} g \in \mathrm{GL}(3,\mathbb{R})$. We have

$$\det(\lambda_3^{-1} g) = (\lambda_3^{-1})^3 \det(g) = 1,$$

hence $\lambda_3^{-1} g \in \mathrm{SL}(3,\mathbb{R})$. We have $\varphi(\lambda_3^{-1} g) = [g]$ and hence φ is surjective, and hence

$$\mathrm{PSL}(3,\mathbb{R}) = \mathrm{PGL}(3,\mathbb{R}).$$

(3) For general n, we still have a function $\varphi\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{PGL}(n,\mathbb{R})$ and we still have that $\ker(\varphi) = Z_{\mathrm{SL}(n,\mathbb{R})}$. For $g \in \mathrm{GL}(n,\mathbb{R})$ we can find a preimage as in (b) exactly when n is odd, since all real numbers then have an n-th root. When n is even, the projection of $g = \mathrm{Diag}(-1, 1, \ldots, 1)$ does not have a preimage, as there is no even power of a real number that is -1. We thus have

$$\mathrm{PSL}(n,\mathbb{R}) \quad \begin{cases} \cong \mathrm{PGL}(n,\mathbb{R}) & \text{if } n \text{ is odd} \\ \ncong \mathrm{PGL}(n,\mathbb{R}) & \text{if } n \text{ is even.} \end{cases}$$

(c) Also for $\mathbb{C}$, one can define $\varphi\colon \mathrm{SL}(n,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$ and calculate its kernel $\ker(\varphi) = Z_{\mathrm{SL}(n,\mathbb{C})}$. Since $\mathbb{C}$ is algebraically closed, every number has an n-th root for every n. Thus the trick in (b) works for all matrices and φ is always surjective, showing the result.