Solution 5

Exercise 1

We consider the Cayley-transformation $r_2 \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}, z \mapsto \frac{z-i}{z+i}$.

(a) Verify that r_2 sends the real line to S^1.

(b) Where does r_2 send S^1?

(c) Draw a picture of how r_2 acts on the Riemann sphere $\hat{\mathbb{C}}$ viewed as a sphere $S^2 \subseteq \mathbb{R}^3$.

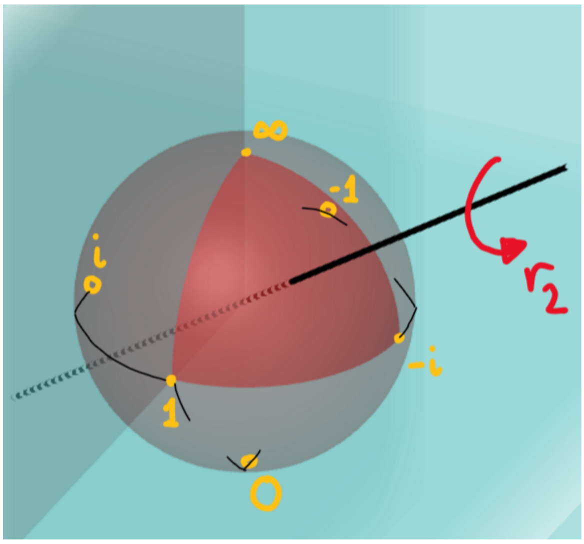
(d) Find an excplicit formula for r_2^{-1}.

Solution:
We know that Möbius transformations (so in particular the Cayley-transformation) send clines to clines. Recall that

$$1 \mapsto -i \mapsto \infty \mapsto 1$$
$$i \mapsto 0 \mapsto -1 \mapsto i$$

(a) The real line is a cline, hence its image under r_2 also has to be a cline. It suffices to consider three points, such as $1, 0, -1$ which get sent to $-i, -1, i$. A cline is uniquely defined by three points and the unique cline going through $-i, -1, i$ is the unit circle S^1.

(b) Since S^1 is a cline, we know that $r_2(S^1)$ is a cline. It suffices to consider three points, such as $1, i$ and $-i$ that get sent to $-i, 0$ and ∞. The unique line through these points is the (vertical) imaginary axis.



(c) .

(d) We use the fact that $r_2^3 = \mathrm{Id}$, to get that $r_2^{-1} = r_2^2$, hence

$$\begin{aligned} r_2^{-1}(z) &= \frac{\frac{z-i}{z+i} - i}{\frac{z-i}{z+i} + i} = \frac{\frac{z-i-zi+1}{z+i}}{\frac{z-i+zi-1}{z+i}} \\ &= \frac{z(1-i)+(1-i)}{z(1+i)-(1+i)} = (-i)\frac{z+1}{z-1} = \frac{-iz-i}{z-1}. \end{aligned}$$

Exercise 2

Show that the subgroup $\mathrm{PSL}(2,\mathbb{R})$ of the orientation preserving Möbius transformations $\mathrm{PSL}(2,\mathbb{C}) \cong \text{Möb}_+$ preserves the upper half plane H.

Solution:
Let $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$. We apply the corresponding Möbius transformation to $z = x + iy$ with $y > 0$. We have

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} z &= \frac{az+b}{cz+d} = \frac{a(x+iy)+b}{c(x+iy)+c} = \frac{ax+b+iay}{cx+d+icy}\frac{cx+d-icy}{cx+d-icy} \\ &= \frac{(ax+b)(cx+d)+acy^2+i(-(ax+b)cy+(cx+d)ay)}{(cx+d)^2+c^2y^2}. \end{aligned}$$

Since the denominator is positive, we just need $-(ax+b)cy + (cx+d)ay$ to be positive. We can use $y > 0$ and $ad - bc = 1$ to get

$$((cx+d)a - (ax+b)c)y = cax + ad - cax - bc)y = y > 0.$$

Exercise 3

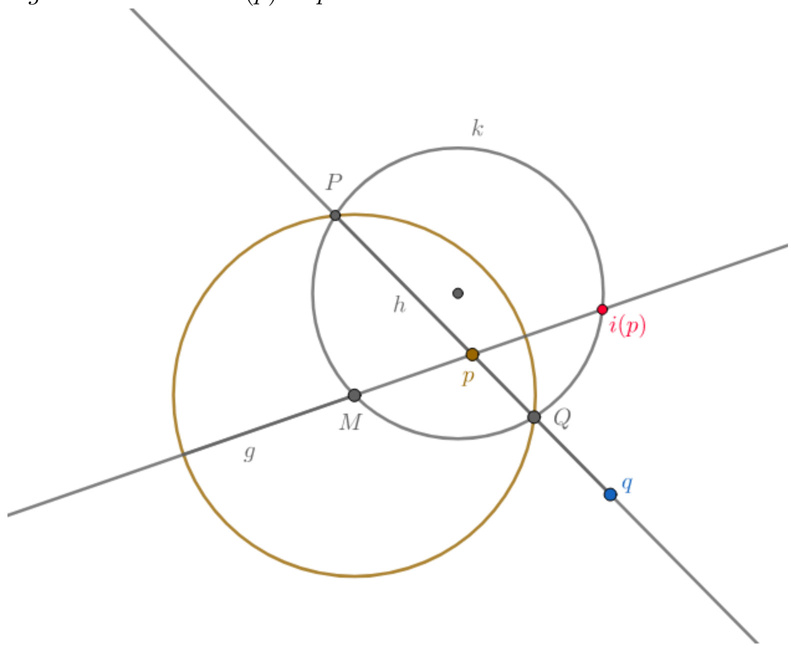
Let $p \in B \setminus \{0\}$ be a point in the unit disk B. Construct the image of p under inversion in the unit circle using only compass and straightedge.

Hint: Draw two straight lines through p and figure out what the circle inversion does to the two lines.

Solution:

To construct the midpint of the unit circle, we choose three points on the circle and construct the perpendicular bisectors between two them each. These three straight lines intersect in M. Next we draw a straight line g from M through p and know that the circle inversion of p must lie on the inversion of the line g, which is g again.

Now we choose an arbitrary straight line h through p that does not pass through M. Since p lies inside the circle, h cuts the circle in two points, call them P and Q. Since h is a cline, and the circle inversion sends clines to clines, we know that the inversion of h also has to contain P and Q. Another point that is in h is ∞. And we know that ∞ gets sent to M by the circle inversion. Thus the inverson of h is the unique circle k through P, Q and M and we can construct it using circle and straightedge using perpendicular bisectors similarly as we did for M. The intersection of k with g is the inversion $i(p)$ of p.



Exercise 4

Consider the orientation-preserving octahedral group O as a subgroup of $\mathrm{Isom}_+(S^2)$.

(a) How many elements does it have?

(b) List all elements by their order.

Solution:
To calculate the number of elements, we can use the orbit stabilizer theorem which states that whenever a group G acts on a space X, then we have for all points $p \in X$

$$|G| = |\operatorname{Stab}_G(p)| \cdot |\operatorname{Orbit}_G(p)|.$$

In our case, O acts on the sphere S^3. The stabilizer of the north pole (the point corresponding to ∞) contains 4 elements (the identity, and the rotations by $k \cdot 90°$ for $k \in \{1, 2, 3\}$). The orbit of ∞ is $\{\infty, 1, -1, i, -i, 0\}$ and hence contains 6 points. This gives $|O| = 4 \cdot 6 = 24$ elements.

By the Satz vom Fussball[1], all rotations have at least two fixed points, through which the axis of rotation goes. By order, we can state all the elements as:

- The identity
- A rotation by 180° around an axis. The axis could go through one of the three pairs of opposite points, or through one of the 6 pairs of opposite edges of the octahedron. So we have 9 elements of order 2.
- For rotations by 120°, there are always two possibilites, to go clockwise or anticlockwise. The possible axes of rotation go through one of the 4 opposite pairs of faces of the octahedron. This gives 8 elements of order 3.
- The axes going through one of the 3 pairs of opposite points also admit rotations by 90°, clockwise or anticlockwise each, resulting in 6 elements of order 4.

We have enumerated all 24 elements of O.

In the following picture the axes are illustrated: Order two elements use the axes of the first and third type. The order 3 axes use type illustrated in the middle, and the order 4 elements use the first type of axis.