Solutions 6

Exercise 1

(a) Show that Möb_+ acts transitively on the set of clines, i.e. for any two clines ℓ, ℓ' there is $g \in \text{Möb}_+$ with $g\ell = \ell'$.

(b) For $z \in \hat{\mathbb{C}}$, what is the point $z' \in \hat{\mathbb{C}}$ that lies on the opposite side of the Riemann sphere $S^2 \cong \hat{\mathbb{C}}$?

(c) What are the clines in $\hat{\mathbb{C}}$ that correspond to great circles[1] on S^2? Which great circles correspond to lines? For those great circles that correspond to circles, how do their radii depend on their centers?

Solution:

(a) Clines are lines or circles. We first note that we can apply translatios in Möb_+ to send any line to a line that goes through 0. We can then apply rotations (multiplications by a complex number of modulus 1) to send that line to the real axis. Thus using translations and rotations we can send any line to any other line. Similarly we can send any circle to any other, by first translating its center to 0 and then scaling the radius by a multiplication with a positive real number. This shows that any circle can get sent to any other circle. Finally we have orientation preserving Möbius transformations such as $z \mapsto 1/z$ that send circles to lines and lines to circles, which concludes the proof of transitivity.

Alternatively, there will be a theorem in class stating that Möb_+ acts transitively on triples of points in $\hat{\mathbb{C}}$. Since clines are uniquely defined by three points, we get the transitive action on clines as a direct consequence.

(b) The formula is

$$z' = \frac{-1}{\bar{z}}$$

which is reflection along the equator ($z \mapsto 1/\bar{z}$) followed by rotation around the North-South-axis by 180°, which is $z \mapsto -z$.

(c) On S^2, great circles are the sets of points that are perpendicular to a given vector $\vec{p}_0$:

$$\{\vec{p} \in S^2 : \vec{p} \cdot \vec{p}_0 = 0\}.$$

To describe great circles in $\hat{\mathbb{C}}$, we start with a point $z_0 = x_0 + iy_0$ and send it to S^2 with the inverse stereographic projection

$$\tau(z_0) = \frac{1}{|z_0|^2 + 1}\left(2x_0, 2y_0, |z_0|^2 - 1\right)$$

[1] A *great circle* on S^2 is a circle that contains two opposite points

A point $z = x + iy$ lies in the cline corresponding to a great circle whenever $\tau(z) \cdot \tau(z_0) = 0$. We calculate

$$0 = \tau(z) \cdot \tau(z_0) = \frac{1}{(|z|^2 + 1)(|z_0|^2 + 1)} \left(4xx_0 + 4yy_0 + (|z|^2 - 1)(|z_0|^2 - 1)\right)$$

and hence

$$\begin{aligned} 0 &= 4xx_0 + 4yy_0 + (x^2 + y^2 - 1)(|z_0|^2 - 1) \\ &= x^2(|z_0|^2 - 1) + y^2(|z_0|^2 - 1) + 4xx_0 + 4yy_0 - (|z_0|^2 - 1) \end{aligned}$$

When $|z_0|^2 \neq 1$, we can divide by $(|z_0|^2 - 1)$ to get

$$0 = x^2 + y^2 + \frac{4x_0}{|z_0|^2 - 1} x + \frac{4y_0}{|z_0|^2 - 1} y - 1$$

and hence

$$\begin{aligned} \left(x + \frac{2x_0}{|z_0|^2 - 1}\right)^2 + \left(y + \frac{2y_0}{|z_0|^2 - 1}\right)^2 &= \frac{4x_0^2}{(|z_0|^2 - 1)^2} + \frac{4y_0^2}{(|z_0|^2 - 1)^2} + 1 \\ &= \left(\frac{2|z_0|}{|z_0|^2 - 1}\right)^2 + 1 \end{aligned}$$

which is exactly the formula for a circle with center point

$$-\frac{2}{|z_0|^2 - 1} z_0$$

and radius

$$\sqrt{\left(\frac{2|z_0|}{|z_0|^2 - 1}\right)^2 + 1}.$$

We note that as the center point becomes larger, the radius approaches the absolute value of the center point and hence the circle gets closer to 0.

When $|z_0| = 1$, $\tau(z_0)$ lies on the equator, and the great circle goes through the north and south poles. In this case the corresponding shape is a line going through 0.

Exercise 2

Let $t \in \mathbb{R}$ and

$$U_t(z) = \frac{z}{1 + tz}$$

(a) Consider the family of all clines tangent to the imaginary axis at the origin. Prove that U_t takes each member of this family to another one.

Hint: compute $U_t'(0)$ and consider its effect.

(b) Consider the family of all circles tangent to the real axis at the origin.

Prove that U_t preserves each member of this family but slides it along itself (as t varies).

(c) Show these families of circles are perpendicular to one another wherever they meet.

(d) Draw these families. Draw arrows to indicate the motion effected by U_t.

Solution:

(a) We first note that $U_t(0) = 0$. We have

$$U_t'(z) = \frac{(1+tz) - zt}{(1+tz)^2} = \frac{1}{(1+tz)^2}$$

with $U_t'(0) = 1$. This means that locally around 0, the function U_t acts as the identity. Hence a cline that is tangent to the imaginary axis at 0 is again sent to a cline tangent to the imaginary axis at 0.

(b) We note that circles tangent to the real axis are of the form

$$\{z = x + iy\,\mathbb{C} \colon x^2 + (y - y_0)^2 = y_0^2\} = \{z \in \mathbb{C} \colon |z - iy_0| = y_0\}$$

for some $y_0 \in \mathbb{R}$. For such points $z = x + iy$, we have $|z|^2 - 2yy_0 = 0$. We now show that $U_t(z)$ lies in the same circle:

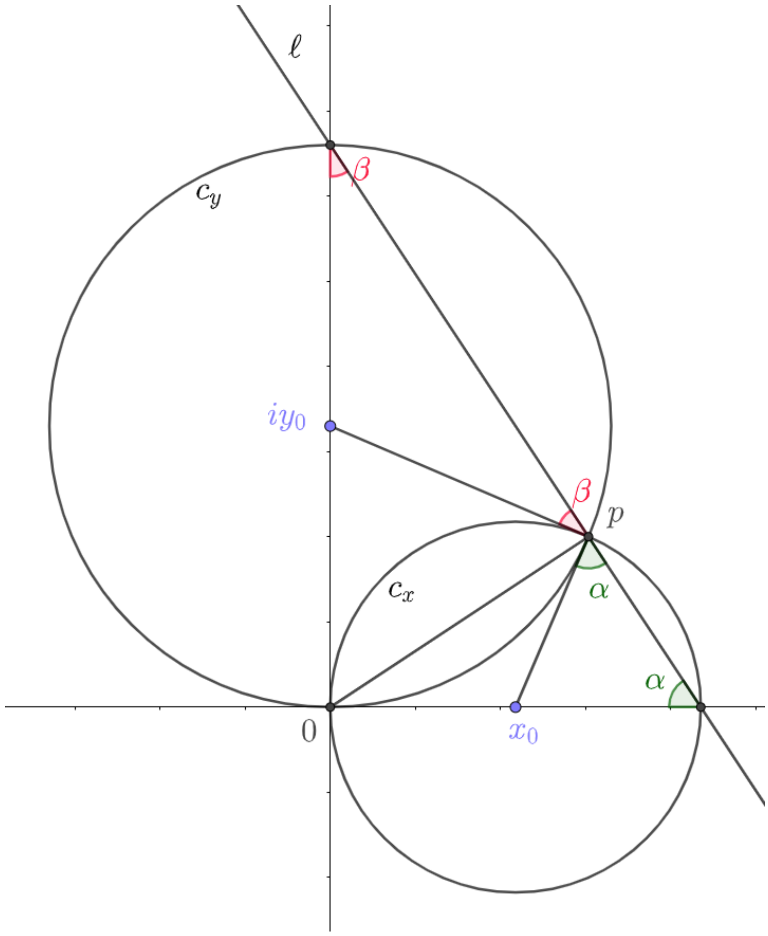
$$\begin{aligned} U_t(z) &= \frac{x+iy}{1+t(x+iy)} = \frac{x+iy}{1+tx+iy}\frac{(1+tx)-iy}{(1+tx)-iy} \\ &= \frac{x(1+tx)+ty^2+i(y(1+tx)-xty)}{(1+tx)^2+(ty)^2} = \frac{x+t|z|^2+iy}{(1+tx)^2+(ty)^2} \end{aligned}$$

If we now plug in $U_t(z)$ into the equation for the circle we get

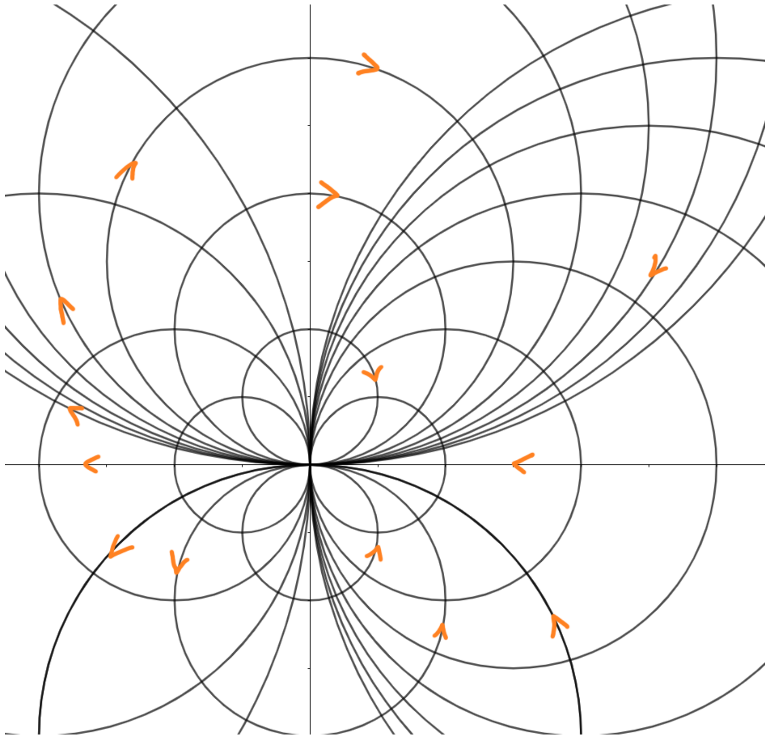
$$\begin{aligned} &\left(\frac{x+t|z|^2}{(1+tx)^2+(ty)^2}\right)^2 + \left(\frac{y}{(1+tx)^2+(ty)^2} - y_0\right)^2 \\ &= \frac{x^2+2xt|z|^2+t^2|z|^4}{((1+tx)^2+(ty)^2)^2} + \frac{y^2}{((1+tx)^2+(ty)^2)^2} - 2\frac{yy_0}{(1+tx)^2+(ty)^2} + y_0^2 \\ &= \frac{1}{((1+tx)^2+(ty)^2)^2}\left(x^2+2xt|z|^2+t^2|z|^4+y^2-2yy_0(1+2tx+t^2|z|^2)\right) + y_0^2 \\ &= \frac{1}{((1+tx)^2+(ty)^2)^2}\left(x^2+y^2-2yy_0+2xt(|z|^2-2yy_0)+t^2|z|^2(|z|^2-2yy_0)\right) + y_0^2 \\ &= y_0^2 \end{aligned}$$

where we used $|z|^2 - 2yy_0 = 0$ three times in the last line. This shows that these circles are preserved under U_t. As U_t depends continuously on t, and $U_t(z) = z$ if and only if $tz = 0$, we see that all the elements of the cricle gets moved continuously along the circle (except for $z = 0$, which is fixed).

(c) This statement follows geometrically. Consider two circles c_x, c_y, with midpoints x_0, iy_0 and radii x_0, y_0 for $x_0, y_0 \in \mathbb{R}$. Let p be the point of intersection of the two circles. Consider The lines from p to $2iy_0$ and from p to $2x_0$. By the Thales theorem, both of these lines have a right angle at p (with respect to the line $\overline{0p}$) and hence these two lines coincide; we will denote this line by ℓ.



We now consider the angle $\alpha = \angle 0, 2x_0, p$ which has the same size as $\angle x_0, p, 2x_0$ and the angle $\beta = \angle 0, 2iy_0, p$ which has the same size as $\angle iy_0, p, 2iy_0$. By the large triangle we know that $\alpha + \beta = 90°$. Now by looking around p we know that the angles between the radii $\overline{x_0 p}$ and $\overline{iy_0 p}$ is $180° - \alpha - \beta = 90°$. Hence the radii are perpendicular to each other and hence the radii are tangents to the other respective circles. So the tangents to the circles are perpendicular at p which is what we mean when we say that the circles are perpendicular at p.



Exercise 3

Show that the set of all inversions in clines generate Möb.

Solution:
We use the fact that Möb is generated by translations T_b, scalar multiplication M_a and the involutions $N\colon z \mapsto 1/z$ and $C\colon z \mapsto \overline{z}$.

We first note that rotations R_φ around 0 by an angle $\varphi \in [0, 2\pi)$ can be constructed from reflections along clines. In fact consider the line $\ell_{\varphi/2}$ which goes through 0 and has an angle $\varphi/2$ to the real axis. Then $R_\varphi = r_{\ell_{\varphi/2}} \circ r_{\ell_0}$, where r_ℓ is the reflection along the line ℓ. (it is enough to check this statement on a basis).

Next, we get translation T_b for $b \in \mathbb{C}$ by considering the reflection along the line ℓ perpendicular to b going through 0 and the parallel line $\ell + b/2$. We have $T_b = r_{\ell+b/2} \circ r_\ell$. (Explicitly, it might be easier to do it by restricting to $b \in \mathbb{R}$ and then use R_φ.)

Similarly, we get for $a \in \mathbb{R}_{>0}$ that M_a can be written as the circle reflection along the unit circle r_{S^1} followed by the circle reflection $r_{\sqrt{a}S^1}$ along the circle of radius $\sqrt{a}$. A multiplication by a complex number $a = re^{i\varphi} \in \mathbb{C} \setminus \{0\}$ is obtained by $M_a = M_r \circ R_\varphi$ and hence can be written as a long composition of reflections along clines.

The involution C is literally the reflection along the real axis, which is a cline. We recall that $r_{S^1} : z \mapsto 1/\bar{z}$ is the reflection along the unit circle. Hence we can obtain $N = r_{S^1} \circ C$. This way we wrote all generators of Möb as compositions of reflections along clines.