Solutions Sheet 7

Exercise 1

For $-1 < t < 1$, consider the Apollonian slide $K_t \colon B_1 \to B_1$ defined by

$$K_t(z) = \frac{z+t}{tz+1} \qquad z \in B_1.$$

(a) Find a t' such that $K_{t'} = K_t \circ K_t$.

(b) A *one-parameter subgroup* in a group G is a function $h \colon \mathbb{R} \to G$, such that

$$h(s+t) = h(s)h(t) \qquad s, t \in \mathbb{R}.$$

Reparametrize the family K_t as a new family $\tilde{K}_t$ which is a one parameter subgroup in $\text{Möb}(B_1)$.

You may want to look up the addition theorems for hyperbolic trigonometric functions.

Solution:
For $t, s \in (-1,1)$, such that $t+s \in (-1,1)$ we have

$$\begin{aligned} K_t \circ K_s(z) &= \frac{\frac{z+s}{sz+1} + t}{t\frac{z+s}{sz+1} + 1} = \frac{z+s+t(sz+1)}{tz+ts+sz+1} \\ &= \frac{(1+ts)z+s+t}{(t+s)z+1+ts} = \frac{z+\frac{s+t}{1+st}}{\frac{s+t}{1+st}z+1} = K_{\frac{s+t}{1+st}}(z) \end{aligned}$$

(a) By the above formula we have $t' = \frac{2t}{1+t^2}$.

(b) When we ask ChatGPT about the addition theorems of hypberbolic trigonometric functions, we get

$$\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}.$$

This suggests that we can choose $h(t) = \tilde{K}_t = K_{\tanh(t)}$ to get

$$\begin{aligned} h(s+t) = \tilde{K}_{s+t} = K_{\tanh(s+t)} &= K_{\frac{\tanh(s)+\tanh(t)}{1+\tanh(s)\tanh(t)}} \\ &= K_{\tanh(s)} \circ K_{\tanh(t)} = \hat{K}_s \circ \hat{K}_t = h(s)h(t) \end{aligned}$$

which shows that $\tilde{K}$ is a one-parameter subgroup of Möb.

Exercise 2

In class, we proved that Möbius transformations on S^2 are conformal by proving it first for affine maps, and then using a generating set of Möb. This exercise investigates an algernative proof using charts.

Let f be a Möbius transformation and $\tilde{f} = \sigma^{-1} \circ f \circ \sigma$ its corresponding map on S^2. We write $\sigma\colon S^2 \to \hat{\mathbb{C}}$ for the stereographic projection centered at the north pole and $\sigma'\colon S^2 \to \hat{\mathbb{C}}$ for the stereographic projection centered at the south pole.

Since $f|_{\mathbb{C}\setminus\{f^{-1}(\infty)\}}$ is holomorphic on $\mathbb{C}\setminus\{f^{-1}(\infty)\}$ and σ is conformal, $\tilde{f}$ is conformal at all points $p \in S^2 \setminus \{N, \tilde{f}^{-1}(N)\}$. We want to show that $\tilde{f}$ is conformal everywhere.

(a) Consider the four maps of $\hat{\mathbb{C}}$

$$\sigma \circ \tilde{f} \circ \sigma^{-1}, \quad \sigma' \circ \tilde{f} \circ (\sigma')^{-1}, \quad \sigma \circ \tilde{f} \circ (\sigma')^{-1}, \quad \sigma' \circ \tilde{f} \circ \sigma^{-1}.$$

Show that for any point $p \in S^2$, at least one of these four maps can be used to prove that $\tilde{f}$ is conformal at p. In this formulation, we don't have to use a generating set.

(b) Why isn't it enough just to consider

$$\sigma \circ \tilde{f} \circ \sigma^{-1}, \quad \sigma' \circ \tilde{f} \circ (\sigma')^{-1}?$$

(c) If we take the approach in (b), what is the set of Möbius transformations for which the proof fails?

Solution:
We already know that for $f_1 = \sigma \circ \tilde{f} \circ \sigma^{-1} = f$, the points N and $\tilde{f}^{-1}(N)$ are bad, as at these places we cannot use that $f|_{\mathbb{C}\setminus\{f^{-1}(\infty)\}}$ is holomorphic and hence conformal.

For $f_2 = \sigma' \circ \tilde{f} \circ (\sigma')^{-1}$, the bad points are S and $\tilde{f}^{-1}(S)$. We see this by considering the stereographic projection $\sigma'\colon S^2 \to \hat{\mathbb{C}}$. Then $\sigma'(S) = \infty$ and $\sigma'(\tilde{f}^{-1}(S)) = \sigma' \circ (\sigma')^{-1} \circ f_2^{-1} \circ \sigma'(S) = f_2^{-1}(\infty)$. We know that $f_2|_{\mathbb{C}\setminus\{f_2^{-1}(\infty)\}}$ is holomorphic and hence conformal. Since also σ' and $(\sigma')^{-1}$ are conformal, we get that $\tilde{f} = (\sigma')^{-1} \circ f_2 \circ \sigma$ is conformal outside the bad points S and $\tilde{f}^{-1}(S)$.

For $f_3 = \sigma \circ \tilde{f} \circ (\sigma')^{-1}$, the bad points are S and $\tilde{f}^{-1}(N)$. We see this by considering the stereographic projection $\sigma'\colon S^2 \to \hat{\mathbb{C}}$. Then $\sigma'(S) = \infty$ and $\sigma'(\tilde{f}^{-1}(N)) = \sigma' \circ (\sigma')^{-1} \circ f_3^{-1} \circ \sigma(N) = f_3^{-1}(\infty)$. We know that $f_3|_{\mathbb{C}\setminus\{f_3^{-1}(\infty)\}}$ is holomorphic and hence conformal. Since also σ' and σ^{-1} are conformal, we get that $\tilde{f} = \sigma^{-1} \circ f_3 \circ \sigma'$ is conformal outside the bad points S and $\tilde{f}^{-1}(N)$.

For $f_4 = \sigma' \circ \tilde{f} \circ \sigma^{-1}$, the bad points are N and $\tilde{f}^{-1}(S)$. We see this by considering the stereographic projection $\sigma\colon S^2 \to \hat{\mathbb{C}}$. Then $\sigma(N) = \infty$ and $\sigma(\tilde{f}^{-1}(S)) = \sigma \circ \sigma^{-1} \circ f_4^{-1} \circ \sigma'(S) = f_4^{-1}(\infty)$. We know that $f_4|_{\mathbb{C}\setminus\{f_4^{-1}(\infty)\}}$ is holomorphic and hence conformal. Since also σ and $(\sigma')^{-1}$ are conformal, we get that $\tilde{f} = (\sigma')^{-1} \circ f_4 \circ \sigma$ is conformal outside the bad points N and $\tilde{f}^{-1}(S)$.

(a) We know that $N \neq S \in S^2$. Since $\tilde{f}$ is bijective, we also know that $\tilde{f}^{-1}(N) \neq \tilde{f}^{-1}(S)$. Outside these four points, we may use any of the maps to show that $\tilde{f}$ is conformal. Thus for any $p \in S^2$, we know that p is not N or not S and that p is not $\tilde{f}^{-1}(N)$ or $\tilde{f}^{-1}(S)$. The following table states for which cases of p we should use which map.

p is not	N	S
$\tilde{f}^{-1}(N)$	f_1	f_3
$\tilde{f}^{-1}(S)$	f_4	f_2

In all cases we have a function to use. Hence $\tilde{f}$ is conformal everywhere on S^2.

(b) It could be that $p = N = \tilde{f}^{-1}(S)$. In this case we cannot use f_1 or f_2 because p is a bad point of both of them.

(c) The only cases where p is a bad point of f_1 and f_2 is when $p = N = \tilde{f}^{-1}(S)$ or $p = S = \tilde{f}^{-1}(N)$. In the first case, $\tilde{f}(N) = S$ and in the second case $\tilde{f}(S) = N$. So the set of Möbius transformations for which the proof would fail if we only use f_1 and f_2 is

$$\{\tilde{g} \in \text{Möb}(S^2) \colon \tilde{g}(N) = S, \text{ or } \tilde{g}(S) = N\}.$$