Solutions 8

Exercise 1

(a) Find an example of an elliptic Möbius transformation that preserves the unit disk.

(b) Find an example of a hyperbolic Möbius transformation that preserves the unit disk.

(c) Find an example of a parabolic Möbius transformation that preserves the unit disk.

(d) Show that there is no loxodromic Möbius transformation that preserves the unit disk, except for the hyperbolic or elliptic ones.

Solution:

(a) The standard elliptic Möbius transformation $z \mapsto e^{i\theta} z$ for $\theta \in (0, 2\pi)$ preserves the unit disk.

(b) We have to find a hyperbolic transformation that preserves B_1. A hyperbolic transformation is a Möbius transformation that is conjugated to the standard hyperbolic transformation $M_\lambda \colon z \mapsto \lambda z$ for $\lambda > 0$. We note that the standard hyperbolic transformation does not preserve B_1. Note that the standard hyperbolic transformation fixes the points 0 and $\infty \in \hat{\mathbb{C}}$ and sends the real axis to itself. If we find a transformation f, that sends the real axis to the unit circle (and make sure that it preserves the interior), then we have found a transformation to conjugate by to get our hyperbolic element that preserves the unit disk. Luckily we already know, that the Cayley transformation

$$r_2 \colon z \mapsto \frac{z-i}{z+i}$$

sends the real line to the unit circle, (Sheet 5, Exercise 1). Thus the conjugation $r_2 \circ M_\lambda \circ r_2^{-1}$ sends the unit circle to the real line to the real line to the unit circle. The interior point $0 \in B_1$ is sent to i to λi, to $(\lambda - 1)/(\lambda + 1) \in B_1$. By continuity, $r_2 \circ M_\lambda \circ r_2^{-1}$ thus preserves the unit disk and is a hyperbolic transformation of B_1.

To calculate the exact expression we use the isomorphism $\mathrm{PGL}(2, \mathbb{C}) \cong \text{Möb}_+$. Then

$$\begin{aligned} r_2 \circ M_\lambda \circ r_2^{-1} &= \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda i + i & \lambda i - i \\ \lambda i - i & \lambda i + i \end{bmatrix} \\ &= \begin{bmatrix} \lambda + 1 & \lambda - 1 \\ \lambda - 1 & \lambda + 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\lambda-1}{\lambda+1} \\ \frac{\lambda-1}{\lambda+1} & 1 \end{bmatrix} \end{aligned}$$

which is just the Apollonian slide

$$K_t \colon z \mapsto \frac{z+t}{tz+1}$$

for

$$t = \frac{\lambda - 1}{\lambda + 1}.$$

(c) The standard parabolic Möbius transformation is $T_b \colon z \mapsto z + b$ for $b \in \mathbb{C} \setminus \{0\}$. There is no $b \in \mathbb{C} \setminus \{0\}$ that preserves the unit disk, so we will have to find a conjugation. Taking $b \in \mathbb{R} \setminus \{0\}$, T_b preserves the real axis. So we can use the Cayley transformation again to conjugate T_b into a Möbius transformation that preserves the unit circle and then the unit disk. As in (b) we get the parabolic element $r_2 \circ T_b \circ r_2^{-1}$ that preserves the unit disk.

To calculate the exact expression we again use the isomorphism $\mathrm{PGL}(2, \mathbb{C}) \cong \text{Möb}_+$. Then

$$\begin{aligned} r_2 \circ M_\lambda \circ r_2^{-1} &= \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda i + i & \lambda i - i \\ \lambda i - i & \lambda i + i \end{bmatrix} \\ &= \begin{bmatrix} 2i - b & b \\ -b & 2i + b \end{bmatrix} \end{aligned}$$

which corresponds to the Möbius transformation

$$z \mapsto \frac{(2i - b)z + b}{-bz + (2i + b)}.$$

(c) The standard loxodromic element is of the form $M_a \colon z \mapsto az$ with $a \in \mathbb{C} \setminus \{0\}$. If $|a| = 1$, then M_a is elliptic. If $a \in \mathbb{R}$, then M_a is hyperbolic. We notice that if the loxodromic element M_a is neither elliptic nor hyperbolic, then it does not fix a cline. Intuitively this is because its fixed points sets (its orbits) are spirals. Let us prove it rigorously: Let C be a circle with center z_0 and radius r, then the image of C under M_a

$$\begin{aligned} M_a(C) &= M_a(\{z \in \mathbb{C} \colon |z - z_0| = r\}) = \{az \in \mathbb{C} \colon |z - z_0| = r\} \\ &= \{z \in \mathbb{C} \colon |z/a - z_0| = r\} = \{z \in \mathbb{C} \colon |z - az_0| = |a|r\} \end{aligned}$$

is a circle of radius $|a|r \neq r$. Thus, when M_a is not elliptic, then $|a| \neq 1$ and M_a does not fix the circle C. Let L be a line given by $L = \{z \in \mathbb{C} \colon z = z_0 + t \cdot v\}$ for $z_0, v \in \mathbb{C}, t \in \mathbb{R}$. Here v is the direction vector of the line. Then

$$\begin{aligned} M_a(L) &= \{az \in \mathbb{C} \colon z = z_0 + tv\} = \{z \in \mathbb{C} \colon z/a = z_0 + tv\} \\ &= \{z \in \mathbb{C} \colon z = az_0 + tav\}. \end{aligned}$$

which is a line with direction vector $av \in \mathbb{C}$. If M_a is not hyperbolic, then $a \notin \mathbb{R}$, hence v and av point in different directions, hence M_a does not preserve the line L. Alltogether we saw that M_a does not preserve cirles or lines, hence it does not preserve a cline.

Since the standard loxodromic element M_a preserves no clines, also no conjugate $r^{-1} \circ M_a \circ r$ preserves a cline, if C were a cline preserved by $r^{-1} \circ M_a \circ r$, then $r(C)$ would be a cline being preserved by M_a. By continuity, any loxodromic element preserving the unit disk also has to preserve the boundary of the unit disk, which is a cline. This is not possible, hence there is no loxodromic element preserving the unit disk (except for the elliptic and hyperbolic special cases).

Exercise 2

(a) Find the stabilizer[1] of 0 in $\text{Möb}(B_1)$.

(b) For a point $x \in B_1$, describe the stabilizer of x in $\text{Möb}(B_1)$ in terms of the group found in (a).

(c) How many elements of $\text{Möb}(B_1)$ fix two points $x \neq y \in B_1$?

Solution:

(a) If

$$f: z \mapsto \frac{az+b}{cz+d}$$

is a map with $f(0) = 0$, then $b = 0$. We know from the lecture that the general form of an orientation preserving Möbius transformation preserving the unit disk is

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}},$$

hence f is of the form

$$f: z \mapsto \frac{az}{\bar{a}} = \frac{a^2}{a\bar{a}} z = \frac{a^2}{|a|^2} z.$$

Since $a^2/|a|^2 \in S^1$, this is a rotation. We conclude that $\text{Stab}_{\text{Möb}_+(B_1)}(0) = \{z \mapsto bz : |b| = 1\} \cong \text{PSO}(2)$.

If we allow orientation reversing Möbius transformation we get

$$\text{Stab}_{\text{Möb}(B_1)}(0) = \{z \mapsto bz : |b| = 1\} \cup \{z \mapsto b\bar{z} : |b| = 1\}.$$

(b) We use the fact that for all $g \in \text{Möb}_+(B_1)$,

$$g \, \text{Stab}_{\text{Möb}(B_1)}(0) g^{-1} = \text{Stab}_{\text{Möb}(B_1)}(g(0))$$

By the transitivity of the group action on B_1, every element $x \in B_1$ can be written as $g(0) = x$ for some $g \in \text{Möb}(B_1)$. Hence

$$\text{Stab}_{\text{Möb}(B_1)}(x) = g^{-1} \, \text{Stab}_{\text{Möb}(B_1)}(0) g.$$

[1] The *stabilizer* of a point $x \in X$ in a group G acting on X is the subgroup $\{g \in G : g(x) = g\}$.

(c) We may restrict ourselves first to the case that $g \in \text{Möb}(B_1)$ fixes 0 and $x \in B_1$. Then if g is orientation preserving, $g(z) = bz$ with $|b| = 1$ and $g(x) = x$, hence $bx = x$, hence $b = 1$ and $g = \text{Id}$. If g is orientation reversing, then $g(z) = b\bar{z}$ with $|b| = 1$ and $g(x) = x$, hence $b\bar{x} = x$, hence

$$b = \frac{x}{\bar{x}} = \frac{x^2}{|x|^2}.$$

Denote by g_x the map $z \mapsto (x/|x|)^2\bar{z}$. We have that $\text{Stab}_{\text{Möb}(B_1)}(\{0, x\}) = \{\text{Id}, g_x\}$ and only contains two elements.

Now in the general case, when we have two points $x, y \in B_1$, then we can find a $g \in \text{Möb}(B_1)$ such that $g(x) = 0$ by transitivity. We then apply the above argument to see that $\text{Stab}_{\text{Möb}(B_1)}(\{0, g(y)\})$ has two elements. Since

$$\text{Stab}_{\text{Möb}(B_1)}(\{x, y\}) = g^{-1}\,\text{Stab}_{\text{Möb}(B_1)}(\{0, g(y)\})g,$$

$\text{Stab}_{\text{Möb}(B_1)}(\{x, y\})$ also has two elements.

Exercise 3

Let $0 < t < 1$ and K_t the Apollonian slide defined by

$$K_t(z) = \frac{z + t}{tz + 1}.$$

(a) Show that for all $s \in (-1, 1)$, $K_t(s) > s$.

(b) Show that for all $z \in B_1$, $\lim_{n\to\infty} K_t^n(z) = 1$ and $\lim_{n\to-\infty} K_t^n(z) = -1$. *Hint: It may help to see K_t as a one-parameter subgroup as in Sheet 7, Exercise 1.*

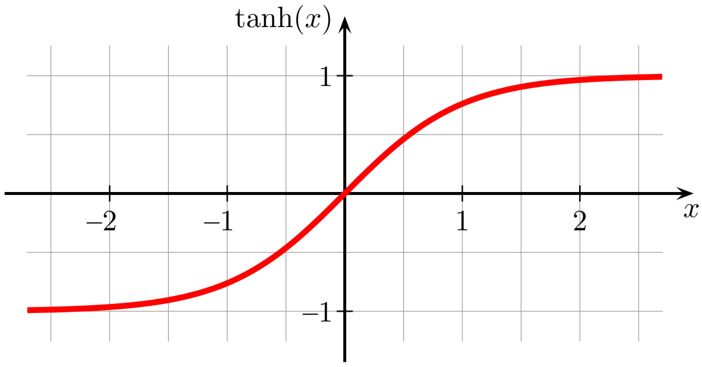
Solution:

(a) Since $|s| < 1$, also $s^2 < 1$. Since $t > 0$, then $ts^2 < t$, hence $ts^2 + s < t + s$, i.e. $s(st + 1) < s + t$ and since $st + 1 > 0$

$$K_t(s) = \frac{s + t}{st + 1} > s.$$

(b) We use the one-parameter description $\tilde{K}_t = K_{\tanh(t)}$. Note that $\tanh(\mathbb{R}_{>0}) \subseteq (0, 1)$ and $\lim_{t\to\infty} \tanh(t) = 1$, as can be seen from the graph of tanh below, or by doing analysis using the definition

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$



We then have

$$K_t^n = \tilde{K}^n_{\operatorname{atanh}(t)} = \tilde{K}_{n\cdot\operatorname{atanh}(t)} = K_{\tanh(n\cdot\operatorname{atanh}(t))}$$

and as $n \to \infty$, $\tanh(n \cdot \operatorname{atanh}(t)) \to 1$, so

$$\lim_{n\to\infty} K_t(z) = \lim_{s\to 1} K_s(z) = \lim_{s\to 1} \frac{z+s}{sz+1} = 1.$$

Similarly, as $n \to -\infty$, $\tanh(n \cdot \operatorname{atanh}(t)) \to -1$ and thus

$$\lim_{n\to-\infty} K_t(z) = \lim_{s\to -1} K_s(z) = \lim_{s\to -1} \frac{z+s}{sz+1} = \frac{z-1}{-z+1} = -1.$$