Solutions 9

Exercise 1

Let $s \leq t \in (-1,1)$ and define

$$\operatorname{cr}(s,t) := \operatorname{cr}(t,s) := \frac{(t+1)(1-s)}{(s+1)(1-t)}.$$

(a) Explain how cr is related to the cross ratio $[z_1, z_2; z_3, z_4]$ in class.

(b) Show that $\operatorname{cr}(s,t) \geq 1$ and $\operatorname{cr}(s,t) = 1$ if and only if $t = s$.

(c) Show that for all $s,t,r \in (-1,1)$, $\operatorname{cr}(s,r) \leq \operatorname{cr}(s,t)\operatorname{cr}(t,r)$.

(d) In view of (a) and (b), how could the cross-ratio cr be used to define a metric on the real interval $(-1,1)$.

(e) Check that the Apollonian slide $K_t \colon z \mapsto \frac{z+t}{tz+1}$ for $t \in (-1,1)$ is an isometry of $(-1,1)$ with the distance from (d), meaning $\forall x,y \in (-1,1), d(x,y) = d(K_t(x), K_t(y))$.

The metric defined on the subset of B_1 will be expanded to the hyperbolic metric on all of B_1.

Solution:

(a) We have $\operatorname{cr}(s,t) = [t, s; -1, 1]$.

(b) Since $s \leq t$, we have $(s+1) \leq (t+1)$ and $(1-t) \leq (1-s)$, hence

$$\frac{(t+1)}{(s+1)} \geq 1 \quad \text{and} \quad \frac{(1-s)}{(s-t)}$$

and $\operatorname{cr}(s,t) \geq 1$. Equality holds if and only if $s = t$.

(c) We first notice the following equality when $s \leq t \leq r$,

$$\operatorname{cr}(s,t)\operatorname{cr}(t,r) = \frac{(s+1)(1-t)}{(t+1)(1-s)} \cdot \frac{(t+1)(1-r)}{(r+1)(1-t)} = \frac{(s+1)(1-r)}{(r+1)(1-s)} = \operatorname{cr}(s,r).$$

This equation shows that the inequality in (b) holds when t lies between s and r. When t is not between s and r, we proceed as follows.

If for instance $t \leq s \leq r$. Then use (a) and the above equality to conclude

$$\operatorname{cr}(s,r) \leq \operatorname{cr}(s,t)\operatorname{cr}(t,s)\operatorname{cr}(s,r) = \operatorname{cr}(s,t)\operatorname{cr}(t,r).$$

(d) (a) reminds us of positive definiteness and (b) reminds us of the triangle inequality, but both of these are multiplicative. We can apply the logarithm to turn multiplication into addition and to obtain a distance. We define the distance between $s, t \in (-1, 1)$ to be

$$d(s, t) := \log \operatorname{cr}(s, t).$$

This function d is positive definite due to (a), symmetric due to the definition of cr and satisfies the triangle inequality due to (b).

(e) We show that K_t preserves the cr. Assume without loss of generality that $x < y$. Then $K_t(x) < K_t(y)$.

$$\begin{aligned}
\operatorname{cr}(K_t(x), K_t(y)) &= \frac{\left(\frac{y+t}{ty+1} + 1\right)\left(1 - \frac{x+t}{xt+1}\right)}{\left(\frac{x+t}{tx+1} + 1\right)\left(1 - \frac{y+t}{yt+1}\right)} \\
&= \frac{(y + t + ty + 1)\,(xt + 1 - x - t)}{(x + t + tx + 1)(yt + 1 - y - t)} \\
&= \frac{(y + 1)(1 + t)(x - 1)(t - 1)}{(x + 1)(1 + t)(y - 1)(t - 1)} = \operatorname{cr}(x, y).
\end{aligned}$$

Alternatively we could have used the fact that all Möbius transformation preserve the cross ratio and $K_t(-1) = -1, K_t(1) = 1$ together with (a).

Since K_t preserves cr, it also preserves $d = \log \circ \operatorname{cr}$.

Exercise 2

(a) Identify $\text{Möb}(B_1)$ with the set of all injective maps of the set $\{0, 1, 2\}$ into S^1.

(b) Show that $\text{Möb}(B_1)$ is homeomorphic to an open subset of the 3-torus $S^1 \times S^1 \times S^1$. What set is excluded?

(c) The 3-torus has the advantage that you can visualize it. It is a cube with its sides suitably identified. Try to draw a picture of the topology of $\text{Möb}(B_1)$.

(d) Is $\text{Möb}(B_1)$ connected? Is $\mathrm{PSL}(2, \mathbb{R})$ connected? Are they simply connected?

Solution:

(a) We use the theorem from the lecture that states that $\text{Möb}(B_1)$ acts transitively on triples of distinct points in S^1. Moreover, the action of an element of $\text{Möb}(B_1)$ is determined by the image of three points in S^1. Let $p_0 = 1, p_1 = i, p_2 = -1$. Then we can identify

$$\begin{aligned}
\text{Möb}(B_1) &\cong \big\{f\colon \{0, 1, 2\} \to S^1 : f \text{ is injective}\big\} \\
g &\mapsto f\colon (i \mapsto g(p_i)).
\end{aligned}$$

We note that the identification is well defined, because Möbius transformations are injective. The identification is surjective, because $\text{Möb}(B_1)$ acts triple-transitively on S^1, and the identification is injective, because a Möbius transformation $g \in \text{Möb}(B_1)$ is already determined by the values of $g(p_i)$ for $i \in \{0, 1, 2\}$.

(b) We note that

$$\begin{aligned}\text{Möb}(B_1) &\cong \big\{f\colon \{0,1,2\} \to S^1 \colon f \text{ is injective}\big\}\\ &\cong \{(x,y,z) \in S^1 \times S^1 \times S^1 \colon x \neq y \neq z \neq x\}.\end{aligned}$$

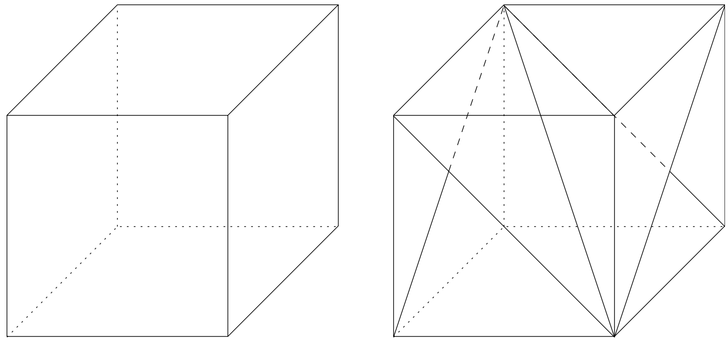
which is an open subset of $S^1 \times S^1 \times S^1$. The set that is excluded is

$$\{(x,y,z) \in S^1 \times S^1 \times S^1 \colon x = y \ \lor \ y = z \ \lor \ x = z\}.$$

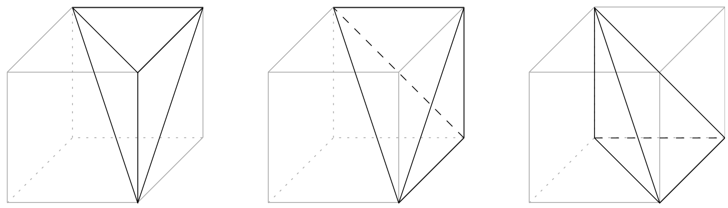
(c) We have

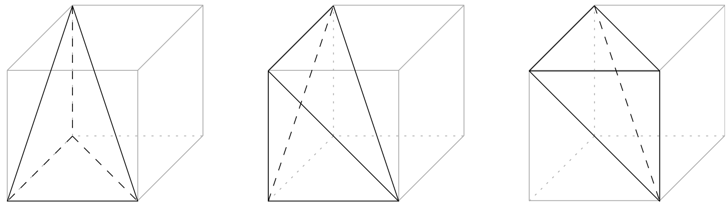
$$S^1 \times S^1 \times S^1 = [0,1]^3/\sim$$

where $(0,y,z) \sim (1,y,z), (x,0,z) \sim (x,1,z)$ and $(x,y,0) \sim (x,y,1)$. The following picture depicts the full torus $S^1 \times S^1 \times S^1$ on the left and the set to be excluded consisting of three planes on the right.

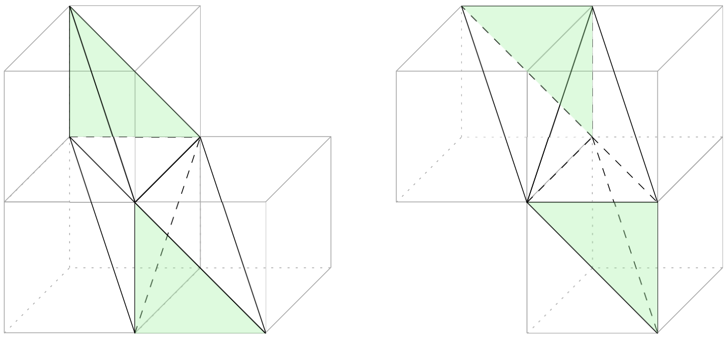


If we cut along the set to be excluded, we get six smaller tetrahaedra, the glueing of the faces of the cubes is still glued together.





When we move the pieces around and start glueing them together where they are supposed to be glued, we see that we obtain two filled in tori. The green sides have to be glued together



(d) From the picture in (c) we see see the topological properties: We see that it has two connected components and each connected component is a filled in torus, hence not simply connected.

From the lecture we know that $\mathrm{PSL}(2, \mathbb{R}) \cong \text{Möb}_+(H_+) \cong \text{Möb}_+(B_1)$. An orientation preserving Möbius transformation g preserving B_1 has to preserve the orientation of the boundary S^1: Looking at the points $1, i, -1$, which come after each other in the mathematically positive direction on S^1, the points $g(1), g(i), g(-1)$ also have to come in this order on S^1. This can be seen by moving a small coordinate system around in B_1.

On the cube $[0,1]^3/\sim$, this means that we only allow elements of the form (x, y, z) with $x < y < z$, $y < z < x$ or $z < x < y$. In the picture of (c), this corresponds exactly to the first filled in torus on the left. Hence $\text{Möb}_+(B_1)$ is connected, but not simply connected.