§27 Triple transitivity of Möb_+ on $\hat{\mathbb{C}}$

The action of the Möbius group is not just transitive, but very transitive.

Theorem 27.1

1) The action of Möb_+ *on* $\hat{\mathbb{C}}$ *is triply transitive, meaning if*

$$z_1, z_2, z_3 \in \hat{\mathbb{C}}$$

is a triple of distinct points, and

$$w_1, w_2, w_3 \in \hat{\mathbb{C}}$$

is another one, then there is a transformation $f \in \text{Möb}_+$ *with*

$$f(z_1) = w_1, \qquad f(z_2) = w_2, \qquad f(z_3) = w_3. \tag{27.1}$$

2) Subject to these conditions, f is unique.

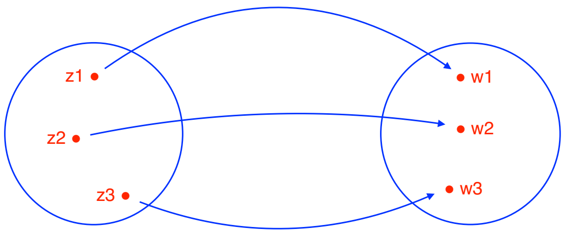

Figure 27.1: Triply transitive

Proof

1. The points z_1, z_2, z_2 and w_1, w_2, w_3 are given. We must find a unique f in Möb_+ that satisfies (27.1).

Let us first assume

$$z_3 = w_3 = \infty.$$

Then f must satisfy

$$f(\infty) = \infty.$$

So f is an affine transformation, and will have the form

$$f(z) = az + b.$$

Since $z_1 \neq z_2$ and $w_1 \neq w_2$, it is geometrically clear that there is an orientation-preserving similarity transformation such that

$$f(z_1) = w_1, \qquad f(z_2) = w_2. \tag{27.2}$$

Just use a for an appropriate scaling and rotation, and b for an appropriate translation.

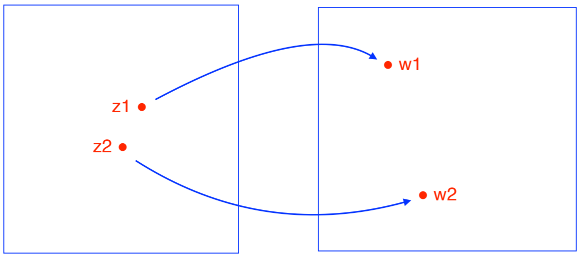

Figure 27.2: The affine group is doubly transitive on $\mathbb{C}$

Algebraically, we get

$$az_1 + b = w_1, \qquad az_2 + b = w_2.$$

Because $z_1 \neq z_2$, these are two independent linear equations for two unknowns (namely a and b), so it has a unique solution, namely

$$a = \frac{w_1 - w_2}{z_1 - z_2}, \qquad b = \frac{z_1 w_2 - z_2 w_1}{z_1 - z_2}.$$

So f exists. These values are forced, so f is unique. We have proven the Theorem under the assumption $z_3 = w_3 = \infty$.

2. Now let z_1, z_2, z_2 and w_1, w_2, w_3 be fully general. We wish to show that there is a unique f in Möb$_+$ such that (27.1) is satisfied.

We know from Theorem 16.1 that Möb$_+$ acts transitively on $\hat{\mathbb{C}}$. So we can select

$$g, h \in \text{Möb}_+$$

such that

$$g(\infty) = z_3, \qquad h(w_3) = \infty.$$

Set

$$k = h \circ f \circ g.$$

Then equation (27.1) is equivalent to

$$k(z_1') = w_1', \qquad k(z_2') = w_2', \qquad k(\infty) = \infty, \tag{27.3}$$

where

$$z_1' = g^{-1}(z_1), \qquad z_2' = g^{-1}(z_2)$$

and

$$w_1' = h(w_1), \qquad w_2' = h(w_2).$$

But by Step 1, equation (27.3) has a unique solution k in Möb$_+$. So equation (27.1) has a unique solution h in Möb$_+$.

□