Sheet 11

Due: To be handed in before 19.05.2023 at 12:00.

1. Exercise

For an integer $n \geq 1$ and $c \in \{0, \dots, n-1\}$, consider the function

$$\beta(\theta) = \sum_{x=c+1}^{n} \binom{n}{x} \theta^x (1-\theta)^{n-x}, \qquad \theta \in (0,1).$$

Note that $\beta(\theta) = \mathbb{P}_\theta(X > c)$ when $X \sim \mathrm{Bin}(n, \theta)$. The goal of this exercise is to show that $\theta \mapsto \beta(\theta)$ is non-decreasing.

(a) Let us denote $p_\theta(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x \in \{0, \dots, n\}$. Show that for any $0 < \theta_1 < \theta_2 < 1$ there exists $x_0 \in \{0, \dots, n-1\}$ such that

$$p_{\theta_2}(x) \begin{cases} \leq p_{\theta_1}(x) & \forall\, x \leq x_0, \\ > p_{\theta_1}(x) & \forall\, x > x_0. \end{cases}$$

(b) Show that $\theta \mapsto \beta(\theta)$ is non-decreasing for any fixed $c \in \{0, \dots, n-1\}$.

2. Exercise

An optical detector can suffer from different sources of inaccuracy. In a given experiment, it was possible to measure the noise level. The following values were observed:

$$\begin{gathered} 1.76, \quad -0.89, \quad 1.04, \quad -3.64, \quad -2.11, \quad 2.73, \quad 0.3, \quad -3.19, \\ -1.24, \quad -1.31, \quad 0.66, \quad -1.58, \quad -4.64, \quad 0.13, \quad -2.96, \quad 0.71. \end{gathered}$$

It is assumed that the noise follows a Gaussian distribution with unknown mean μ and variance σ^2. We want to test

$$H_0 \colon \mu = 0 \qquad \text{versus} \qquad H_1 \colon \mu \neq 0.$$

We take $\alpha = 0.05$.

(a) Construct a suitable test for this problem.

(b) What is your decision? We give:

the 0.95-quantile of $\mathcal{N}(0,1) = 1.64$,

the 0.975-quantile of $\mathcal{N}(0,1) = 1.96$,

the 0.95-quantile of $\mathcal{T}_{15} = 1.75$,

the 0.975-quantile of $\mathcal{T}_{15} = 2.13$.