Sheet 11

Due: To be handed in before 19.05.2023 at 12:00.

1. Exercise

For an integer $n \geq 1$ and $c \in \{0, \dots, n-1\}$, consider the function

$$\beta(\theta) = \sum_{x=c+1}^{n} \binom{n}{x} \theta^x (1-\theta)^{n-x}, \qquad \theta \in (0,1).$$

Note that $\beta(\theta) = \mathbb{P}_\theta(X > c)$ when $X \sim \text{Bin}(n, \theta)$. The goal of this exercise is to show that $\theta \mapsto \beta(\theta)$ is non-decreasing.

(a) Let us denote $p_\theta(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x \in \{0, \dots, n\}$. Show that for any $0 < \theta_1 < \theta_2 < 1$ there exists $x_0 \in \{0, \dots, n-1\}$ such that

$$p_{\theta_2}(x) \begin{cases} \leq p_{\theta_1}(x) & \forall\, x \leq x_0, \\ > p_{\theta_1}(x) & \forall\, x > x_0. \end{cases}$$

(b) Show that $\theta \mapsto \beta(\theta)$ is non-decreasing for any fixed $c \in \{0, \dots, n-1\}$.

Solution:

(a) We have that $\sum_{x=0}^n p_{\theta_1}(x) = \sum_{x=0}^n p_{\theta_2}(x) = 1$, so $\sum_{x=0}^n p_{\theta_1}(x) - p_{\theta_2}(x) = 0$. Thus, $p_{\theta_2} - p_{\theta_1}$ cannot be of a constant sign on $\{0, \dots, n\}$. In other words, there are $x \in \{0, \dots, n\}$ for which $p_{\theta_2}(x)/p_{\theta_1}(x) \leq 1$ and others for which $p_{\theta_2}(x)/p_{\theta_1}(x) > 1$. Call S_- and S_+ the sets of the former and the latter x's. Now,

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \frac{\theta_2^x(1-\theta_2)^{n-x}}{\theta_1^x(1-\theta_1)^{n-x}} = \left(\underbrace{\frac{\theta_2}{\theta_1}\frac{1-\theta_1}{1-\theta_2}}_{>1} \right)^x \left(\frac{1-\theta_2}{1-\theta_1} \right)^n,$$

which means that the function $x \mapsto p_{\theta_2}(x)/p_{\theta_1}(x)$ is strictly increasing on $\{0, \dots, n\}$. Hence, S_- and S_+ have to be of the form $S_- = \{0, \dots, x_0\}$ and $S_1 = \{x_0+1, \dots, n\}$ for some $x_0 \in \{0, \dots, n-1\}$. Note that x_0 depends on θ_1 and θ_2.

(b) Let $0 < \theta_1 < \theta_2 < 1$. We want to show that $\beta(\theta_1) \leq \beta(\theta_2)$. We have

$$\begin{aligned} \beta(\theta_2) - \beta(\theta_1) &= \sum_{x=c+1}^{n} p_{\theta_2}(x) - p_{\theta_1}(x) \\ &= \sum_{x=0}^{x_0} (p_{\theta_2}(x) - p_{\theta_1}(x))\mathbb{1}_{x \geq c+1} + \sum_{x=x_0+1}^{n} (p_{\theta_2}(x) - p_{\theta_1}(x))\mathbb{1}_{x \geq c+1} \end{aligned}$$

with

$$\mathbb{1}_{x \geq c+1} \begin{cases} \leq \mathbb{1}_{x_0 \geq c+1} & \forall\, x \leq x_0, \\ \geq \mathbb{1}_{x_0 \geq c+1} & \forall\, x > x_0, \end{cases}$$

implying that

$$\beta(\theta_2) - \beta(\theta_1) \geq \mathbb{1}_{x_0 \geq c+1} \sum_{x=0}^{x_0} p_{\theta_2}(x) - p_{\theta_1}(x) + \mathbb{1}_{x_0 \geq c+1} \sum_{x=x_0+1}^{n} p_{\theta_2}(x) - p_{\theta_1}(x)$$

since

$$p_{\theta_2}(x) - p_{\theta_1}(x) \begin{cases} \leq 0 & \forall\, x \leq x_0, \\ > 0 & \forall\, x > x_0. \end{cases}$$

Thus,

$$\beta(\theta_2) - \beta(\theta_1) \geq \mathbb{1}_{x_0 \geq c+1} \sum_{x=0}^{n} p_{\theta_2}(x) - p_{\theta_1}(x) = 0,$$

so $\beta(\theta_2) \geq \beta(\theta_1)$.

2. Exercise

An optical detector can suffer from different sources of inaccuracy. In a given experiment, it was possible to measure the noise level. The following values were observed:

$$\begin{aligned} &1.76, \quad -0.89, \quad 1.04, \quad -3.64, \quad -2.11, \quad 2.73, \quad 0.3, \quad -3.19, \\ &-1.24, \quad -1.31, \quad 0.66, \quad -1.58, \quad -4.64, \quad 0.13, \quad -2.96, \quad 0.71. \end{aligned}$$

It is assumed that the noise follows a Gaussian distribution with unknown mean μ and variance σ^2. We want to test

$$H_0\colon \mu = 0 \qquad \text{versus} \qquad H_1 : \mu \neq 0.$$

We take $\alpha = 0.05$.

(a) Construct a suitable test for this problem.

(b) What is your decision? We give:
the 0.95-quantile of $\mathcal{N}(0,1) = 1.64$,
the 0.975-quantile of $\mathcal{N}(0,1) = 1.96$,
the 0.95-quantile of $\mathcal{T}_{15} = 1.75$,
the 0.975-quantile of $\mathcal{T}_{15} = 2.13$.

Solution:

(a) Under H_0, we know that $\sqrt{n}\bar{X}_n/S_n \sim \mathcal{T}_{(n-1)}$ with $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ and $n = 16$. Thus, a suitable test is the following student test

$$\phi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \sqrt{n}|\bar{X}_n|/S_n > t_{n-1,1-\alpha/2}, \\ 0 & \text{otherwise,} \end{cases}$$

where $t_{n-1,1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of $\mathcal{T}_{(n-1)}$.

(b) We compute $\bar{X}_n = -0.889$, $S_n = 2.077$, and $\sqrt{16}|\bar{X}_n|/S_n = 1.712 < 2.13$. We do not reject H_0.