Sheet 12

Due: To be handed in before 26.05.2023 at 12:00.

1. Exercise

When $X_1, \ldots, X_n$ are i.i.d. $\sim \mathcal{N}(\mu, \sigma^2)$ and we want to test

$$H_0: \quad \mu = 0 \qquad \text{versus} \qquad H_1: \quad \mu \neq 0,$$

one could either use the Student-test

$$\Phi_1(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \frac{\sqrt{n}|\bar{X}_n|}{S_n} > t_{n-1,1-\alpha/2}, \\ 0 & \text{otherwise} \end{cases}$$

with $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $t_{n-1,1-\alpha/2}$ the $(1-\alpha/2)$-quantile of $\mathcal{T}_{(n-1)}$ or the sign test

$$\Phi_2(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } |T_n - n/2| > c_\alpha, \\ 0 & \text{otherwise} \end{cases}$$

with $T_n = \sum_{i=1}^n \mathbb{1}_{X_i > 0}$ and c_α as in question (1.c).

(a) Explain why one can use Φ_2 in this testing problem.

(b) Assume that $\sigma = 2$ but this value is still unknown to you. Below, we give the values of the power of Φ_1 and Φ_2 for different sample sizes n and values $\mu \in \Theta_1 = \mathbb{R} \backslash \{0\}$.

$\mu \setminus n$	10	10	20	20	50	50
0.5	0.61	0.04	0.79	0.12	0.95	0.22
-1	0.81	0.13	0.96	0.38	0.99	0.73
2	0.99	0.51	≈ 1	0.91	≈ 1	0.99

In the table, the first, third, and fifth column show $\beta_1(\mu)$ for n =10, 20, respectively 50, and the second, fourth, and sixth column show $\beta_2(\mu)$ for n =10, 20, respectively 50.

It seems that both $\beta_1(\mu)$ and $\beta_2(\mu)$ increase with n and $|\mu|$ and $\beta_1(\mu) > \beta_2(\mu)$. Is this expected?

2. Exercise

Application of the 2-sample student test: Samples of wood were obtained from the core and periphery of a certain Byzantine church. The date of wood was determined, yielding the following data.

core:

1294	1279	1272	1264	1263	1254	1251
1250	1248	1240	1232	1220	1218	1210

periphery:

1284	1272	1256	1254	1242	1274	1264	1255	1250

Use the 2-sample t-test (or student test) of level 0.05 to determine whether the mean age of the core is the same as the mean age of the periphery. We give $t_{21,0.975} = 2.079$.