Sheet 12

Due: To be handed in before 26.05.2023 at 12:00.

1. Exercise

When $X_1, \ldots, X_n$ are i.i.d. $\sim \mathcal{N}(\mu, \sigma^2)$ and we want to test

$$H_0 : \quad \mu = 0 \qquad \text{versus} \qquad H_1 : \quad \mu \neq 0,$$

one could either use the Student-test

$$\Phi_1(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \frac{\sqrt{n}|\bar{X}_n|}{S_n} > t_{n-1,1-\alpha/2}, \\ 0 & \text{otherwise} \end{cases}$$

with $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $t_{n-1,1-\alpha/2}$ the $(1-\alpha/2)$-quantile of $\mathcal{T}_{(n-1)}$ or the sign test

$$\Phi_2(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } |T_n - n/2| > c_\alpha, \\ 0 & \text{otherwise} \end{cases}$$

with $T_n = \sum_{i=1}^n \mathbb{1}_{X_i > 0}$ and c_α as in question (1.c).

(a) Explain why one can use Φ_2 in this testing problem.

(b) Assume that $\sigma = 2$ but this value is still unknown to you. Below, we give the values of the power of Φ_1 and Φ_2 for different sample sizes n and values $\mu \in \Theta_1 = \mathbb{R} \backslash \{0\}$.

$\mu \setminus n$	10	10	20	20	50	50
0.5	0.61	0.04	0.79	0.12	0.95	0.22
-1	0.81	0.13	0.96	0.38	0.99	0.73
2	0.99	0.51	≈ 1	0.91	≈ 1	0.99

In the table, the first, third, and fifth column show $\beta_1(\mu)$ for n =10, 20, respectively 50, and the second, fourth, and sixth column show $\beta_2(\mu)$ for n =10, 20, respectively 50.

It seems that both $\beta_1(\mu)$ and $\beta_2(\mu)$ increase with n and $|\mu|$ and $\beta_1(\mu) > \beta_2(\mu)$. Is this expected?

Solution:

(a) This can be explained by the fact that the expectation μ is also the median of the distribution of $\mathcal{N}(\mu, \sigma^2)$ and that the cdf of $\mathcal{N}(\mu, \sigma^2)$ is continuous at μ. In fact, the density of the distribution of $\mathcal{N}(\mu, \sigma^2)$ is symmetric around μ: $f(\mu + x) = f(\mu - x)$ for all $x \in \mathbb{R}$. Thus,

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\mu} f(t)dt = \int_{-\infty}^{0} f(\mu + x)dx \\ &= \int_{-\infty}^{0} f(\mu - x)dx = \int_{\mu}^{\infty} f(t)dt = 1 - \int_{-\infty}^{\mu} f(t)dt = 1 - F(\mu). \end{aligned}$$

Hence, $F(\mu) = 1/2$. F is continuous on $\mathbb{R}$ because it is $C^1(\mathbb{R})$ with $F' = f$, so F is continuous at μ. F is strictly increasing and, hence, bijective from $\mathbb{R}$ to $(0, 1)$. Thus, $\mu = F^{-1}(1/2)$, where F^{-1} is the inverse of F. This shows that μ is the median of $\mathcal{N}(\mu, \sigma^2)$.

(b) When n increases, we have more information about the unknown distribution and, hence, it is expected that both $\beta_1(\mu)$ and $\beta_2(\mu)$ increase with n for any $\mu \in \Theta_1$.

When $|\mu|$ increases, the evidence against the null hypothesis $H_0 : \ \mu = 0$ becomes stronger and, hence, it is expected that the power increases with $|\mu|$.

It is also expected that Φ_2 is less powerful than Φ_1: Φ_2 does not at all use the information that the data are i.i.d. Gaussian distributed.

2. Exercise

Application of the 2-sample student test: Samples of wood were obtained from the core and periphery of a certain Byzantine church. The date of wood was determined, yielding the following data.

core:

1294	1279	1272	1264	1263	1254	1251
1250	1248	1240	1232	1220	1218	1210

periphery:

1284	1272	1256	1254	1242	1274	1264	1255	1250

Use the 2-sample t-test (or student test) of level 0.05 to determine whether the mean age of the core is the same as the mean age of the periphery. We give $t_{21,0.975} = 2.079$.

Solution:

The 2-sample t-test is

$$\Phi(X_1, \dots, X_n, Y_1, \dots, Y_m) = \begin{cases} 1 & \text{if } \sqrt{\frac{nm}{n+m}} \frac{|\bar{X}_n - \bar{Y}_m|}{S_{n,m}} > t_{n+m-2,1-\alpha/2}, \\ 0 & \text{otherwise,} \end{cases}$$

where $t_{n+m-2,1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of $\mathcal{T}_{(n+m-2)}$ and

$$S^2_{n,m} = \frac{1}{n+m-2} \left(\sum_{i=1}^{m} (X_i - \bar{X}_n)^2 + \sum_{j=1}^{m} (Y_j - \bar{Y}_m)^2 \right).$$

Here: $n = 14$, $X_1, \dots, X_n$ are the dates for the wood taken from the core; $m = 9$, $Y_1, \dots, Y_n$ are the dates for the wood taken from the periphery. The underlying assumption is that $X_1, \dots, X_n$ are i.i.d. $\sim \mathcal{N}(\mu_1, \sigma^2)$, and $Y_1, \dots, Y_m$ are i.i.d. $\sim \mathcal{N}(\mu_2, \sigma^2)$, and $(X_1, \dots, X_n) \perp\!\!\!\perp (Y_1, \dots, Y_m)$. We want to test

$$H_0: \quad \mu_1 = \mu_2 \qquad \text{versus} \qquad H_1: \quad \mu_1 \neq \mu_2.$$

We have $\bar{X}_n = 1249.64$ and $\bar{Y}_m = 1261.22$. Then,

core-$\bar{X}_n$:

44.36	29.36	22.36	14.36	13.36	4.36	1.36
0.36	-1.64	-9.64	-17.64	-29.64	-31.64	-39.64

periphery-$\bar{Y}_m$:

22.78	10.78	-5.22	-7.22	-19.22	12.78	2.78	-6.22	-11.22

and $S^2_{n,m} = 429.1795$. Thus,

$$|T_{n,m}| := \sqrt{\frac{nm}{n+m}} \frac{|\bar{X}_n - \bar{Y}_m|}{S_{n,m}} = \sqrt{\frac{14 \times 9}{23}} \frac{|1249.64 - 1261.22|}{\sqrt{429.1795}} = 1.308.$$

Since $|T_{n,m}| < t_{21,0.975}$, we do not have enough evidence against H_0.