Sheet 1

Due: To be handed in before 03.03.2023 at 12:00.

1. Exercise

Let A, B, C be three events. Express the following events using A, B, C and the operations $\cap$, $\cup$, ^{c} (complement of a set).

E_1 = "At least one of the events A, B, C occurs."
E_2 = "At most one of the events A, B, C occurs."
E_3 = "Exactly one of the events A, B, C occurs."
E_4 = "B can only occur when A or C occurs."
E_5 = "If A occurs, then B occurs as well."
E_6 = "At least one of the events A, B, C occurs, but not all of them at the same time."

Solution:

1. $$E_1 = A \cup B \cup C.$$

2. We can express E_2 as: E_2 = "It cannot happen that two events occur at the same time." Thus,

$$E_2 = [(A \cap B) \cup (A \cap C) \cup (B \cap C)]^c = (A^c \cup B^c) \cap (A^c \cup C^c) \cap (B^c \cup C^c).$$

Alternatively, by the same logic: E_2 = "At least two events must not occur at the same time." Thus,

$$E_2 = (A^c \cap B^c) \cup (A^c \cap C^c) \cup (B^c \cap C^c).$$

<u>Remark:</u> We can transition from one set-theoretic expression of E_2 to the other one by using

$$\begin{aligned}(A^c \cap B^c) \cup (A^c \cap C^c) \cup (B^c \cap C^c) &= (A \cup B)^c \cup (A \cup C)^c \cup (B \cup C)^c \\ &= [(A \cup B) \cap (A \cup C) \cap (B \cup C)]^c\end{aligned}$$

and

$$(A \cap B) \cup (A \cap C) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \cap (B \cup C).$$

The latter holds because:

$$\begin{aligned}(A \cup B) \cap (A \cup C) \cap (B \cup C) &= (A \cup B) \cap ([A \cap (B \cup C)] \cup [C \cap (B \cup C)]) \\ &= (A \cup B) \cap [(A \cap B) \cup (A \cap C) \cup C] \\ &= [(A \cup B) \cap A \cap B] \cup [(A \cup B) \cap C] \\ &= (A \cap B) \cup (A \cap C) \cup (B \cap C).\end{aligned}$$

3. $$E_3 = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C).$$

4. We express E_4 as "B can only occur when $A \cup C$ occurs" and, hence,

$$E_4 = [B \cap (A \cup C)^c]^c = B^c \cup (A \cup C) = A \cup B^c \cup C.$$

5.
$$E_5 = (A \cap B^c)^c = A^c \cup B.$$

6.
$$E_6 = (A \cup B \cup C) \cap (A \cap B \cap C)^c.$$

2. Exercise

We throw a green and a red die and consider the following events.

$E_1 =$ "None of the numbers is bigger than 2."
$E_2 =$ "The numbers are equal."
$E_3 =$ "The number on the red die is twice the number on the green die."
$E_4 =$ "The number on the red die is exactly one smaller or one bigger than the number on the green die."
$E_5 =$ "If the number on the red die is at most 5, then the number on the green die is equal to 6."

(a) Write down the sample space Ω for this random experiment and express the above events as subsets of Ω.

(b) Which one of the above events remains unchanged if we do not know anymore the color of the dice?

Solution:

(a) The sample space is $\Omega = \{(i, j) \colon 1 \leq i \leq 6,\ 1 \leq j \leq 6\}$, where i is the number on the red die and j is the number on the green die (or reverse). In other terms,

$$\Omega = \{1, \ldots, 6\} \times \{1, \ldots, 6\} = \{1, \ldots, 6\}^2.$$

Thus, we obtain for the events $E_1, \ldots, E_5$:

$E_1 = \{(1,1); (1,2); (2,1); (2,2)\}$
$E_2 = \{(1,1); (2,2); (3,3); (4,4); (5,5); (6,6)\}$
$E_3 = \{(2,1); (4,2); (6,3)\}$
$E_4 = \{(1,2); (2,1); (2,3); (3,2); (3,4); (4,3); (4,5); (5,4); (5,6); (6,5)\}$
$E_5 = \{(1,6); (2,6); (3,6); (4,6); (5,6); (6,1); (6,2); (6,3); (6,4); (6,5); (6,6)\}$

(b) That one of the events remains unchanged when we forget the color of the dice is equivalent to the fact that the event is expressed by the same subset of Ω if the dice were to change colors. This is the case if the subset is symmetric under changing the i and the j component. This is the case for E_1, E_2, E_4 and E_5.

3. Exercise

Consider an urn containing N numbered balls, K of which are red and $N - K$ are white. Without loss of generality, we may assume that the balls with numbers $1, 2, \ldots, K$ are red. Now we draw <u>without</u> replacement n balls from the urn ($n \leq N$).

(a) What is the sample space Ω that corresponds to this random experiment?

(b) Find the cardinality $|\Omega|$ of Ω.

(c) Consider the event $R_k = \{\text{there are exactly } k \text{ red balls in the sample}\}$. Find the cardinality $|R_k|$ for every $k \in \{0, \ldots, K\}$.

(d) Compute the ratio $|R_k|/|\Omega|$. Assuming that $K, N \to \infty$ and $K/N \to p$ for a $p \in (0,1)$, to which limit does $|R_k|/|\Omega|$ converge?

Solution:

(a) ω_1 denotes the first ball we draw, ω_2 the second etc. Then

$$\Omega = \{(\omega_1, \dots, \omega_n)\colon 1 \le \omega_i \ne \omega_j \le N\}.$$

(b) When we draw the first time, there are still N balls in the urn, so there are N possibilities for ω_1. After having drawn the first ball, $N-1$ balls remain in the urn since we draw without replacement. Thus, there are $N-1$ possibilities for ω_2, then $N-2$ possibilities for ω_3 etc. In particular,

$$|\Omega| = N(N-1)\cdot \ldots \cdot (N-n+1) = \frac{N!}{(N-n)!}.$$

(c) We can write R_k as the set

$$R_k = \left\{ (\omega_1, \dots, \omega_n) \in \Omega\colon \left[\begin{array}{c} \text{there exist } 1 \le i_1 < \cdots < i_k \le n \text{ such that } \omega_{i_1}, \dots, \omega_{i_k} \in \{1, \dots, K\} \\ \text{and } \omega_i \in \{K+1, \dots, N\} \text{ for all } i \in \{1, \dots, n\} \backslash \{i_1, \dots, i_k\} \end{array} \right] \right\}.$$

Let us first fix the scenario where $i_1 = 1$, ..., $i_k = k$. In this scenario, the first k balls we draw are red and the remaining $n-k$ are all white. How many cases do we have in this case? Upon drawing the first time, there are still all K red balls in the urn, so there are K possibilities for $\omega_{i_1} = \omega_1$. As in question (b), there are $K-1$ possibilities for $\omega_{i_2} = \omega_2$ etc. until there are $K-k+1$ possibilities for $\omega_{i_k} = \omega_k$. Now, all white balls are still in the urn, so for ω_{k+1} there are $N-K$ possibilities. Analogously there remain $N-K-1$ possibilities for ω_{k+2} and at the end we are left with $N-K-(n-k)+1$ possibilities for ω_n. In total, in the scenario $i_1 = 1$, ..., $i_k = k$, there are this many possibilities:

$$K(K-1)\cdot \ldots \cdot (K-k+1)(N-K)(N-K-1)\cdot \ldots \cdot (N-K-(n-k)+1).$$

Now we have to count how many possible $(i_1, \dots, i_k)$ there are. That means we have to count the number of subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, where these subsets contain k distinct elements each. There are $\binom{n}{k}$-many such subsets. Given such a subset, the possibilities for $\omega_{i_1}, \dots, \omega_{i_k}$ are analogous to the first case we considered. In total, we found

$$\begin{aligned} |R_k| &= \binom{n}{k} K(K-1)\cdot \ldots \cdot (K-k+1)(N-K)(N-K-1)\cdot \ldots \cdot (N-K-(n-k)+1) \\ &= \binom{n}{k} \frac{K!}{(K-k)!} \frac{(N-K)!}{(N-K-(n-k))!} \end{aligned}$$

(d) The ratio is

$$\frac{|R_k|}{|\Omega|} = \frac{\frac{n!}{k!(n-k)!}\frac{K!}{(K-k)!}\frac{(N-K)!}{(N-K-(n-k))!}}{\frac{N!}{(N-n)!}} = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}},$$

where $0 \le k \le K$ and $0 \le n \le N$.

<u>Remark:</u> We can regard $|R_k|/|\Omega|$ as the probability of the event R_k computed under the assumption that all elements $(\omega_1, \dots, \omega_n) \in \Omega$ are equi-probable (have the same probability). The probability $\binom{K}{k}\binom{N-K}{n-k}/\binom{N}{n}$ describes the so-called hypergeometric law with parameters K, N and and sample size n.

Now we consider the case with K, N large such that $K/N \to p \in (0,1)$. We compute:

$$\begin{aligned}
\frac{|R_k|}{|\Omega|} &= \binom{n}{k} \frac{K(K-1)\cdot \ldots \cdot (K-k+1)}{N(N-1)\cdot \ldots \cdot (N-k+1)} \frac{(N-K)(N-K-1)\cdot \ldots \cdot (N-K-(n-k)+1)}{(N-k)(N-k-1)\cdot \ldots \cdot (N-n+1)} \\
&= \binom{n}{k} \left(\frac{K}{N}\right)^k \prod_{i=0}^{k-1} \left(\frac{1-\frac{i}{K}}{1-\frac{i}{N}}\right) \prod_{i=0}^{n-k-1} \left(\frac{1-\frac{K+i}{N}}{1-\frac{k+i}{N}}\right) \\
&\xrightarrow[K/N \to p]{K,N\to\infty} \binom{n}{k} p^k \prod_{i=0}^{k-1} \left(\frac{1-0}{1-0}\right) \prod_{i=0}^{n-k-1} \left(\frac{1-p}{1-0}\right) \\
&= \binom{n}{k} p^k (1-p)^{n-k}.
\end{aligned}$$

We recognize the binomial law with parameters p, n.